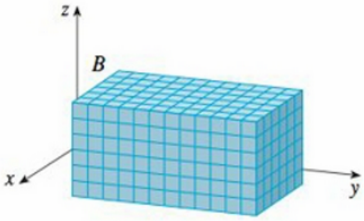
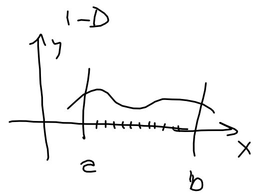
15.6 TRIPLE INTEGRALS





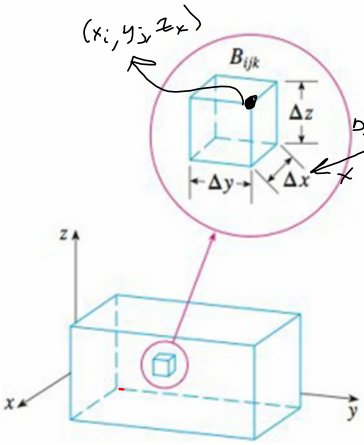
1-D: Break up an interval into subintervals.

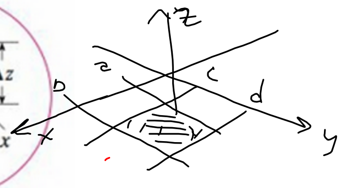
2-D: Break up a rectangle into subrectangles.

3-D: Break up a box into sub-boxes.

$$B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$





The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists. $\quad \Delta V = \Delta x\, \Delta y\, \Delta z.$

We can simplify the writing of this as follows:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

if we choose $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) = (x_i, y_j, z_k)$

FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx = \ldots$$

Clairaut for Integrals.

xyz, xzy, yxz, yzx, zxy, zyx

There are P(3,3) ways to permute the order of integration: That means 3*2*1 = 6 ways.

The triple integral over a general bounded region E

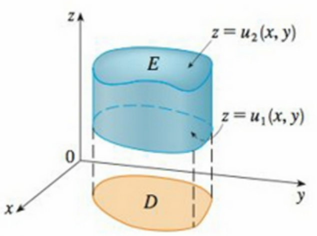


FIGURE 2
A type 1 solid region

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Now for Type I and Type II projections beneath the Type 1 solid:

* Roman Numerals for projection

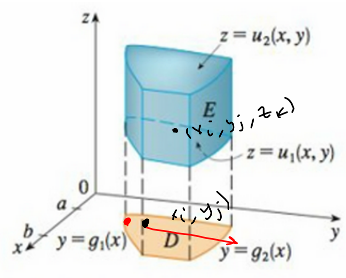


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

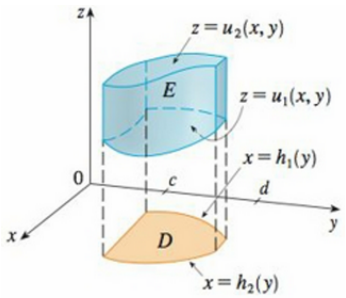


FIGURE 4
A type 1 solid region with a type II projection

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

dydz OR dzdy

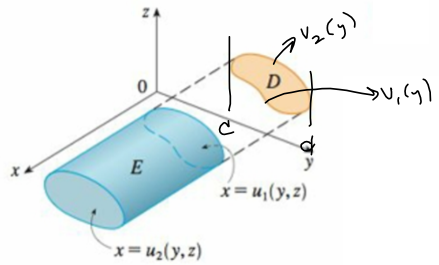
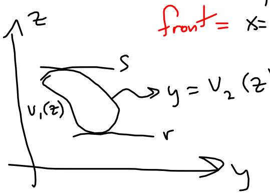


FIGURE 7 A type 2 region TYPE I

$$\int_c^d \int_{z=v_1(y)}^{v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} dx\, dz\, dy$$

Type II

$$\int_r^s \int_{v_1(z)}^{v_2(z)} \int_{x=u_1(y,z)}^{x=u_2(y,z)}$$



A **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

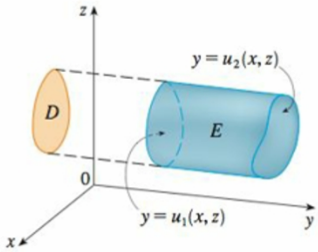
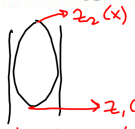
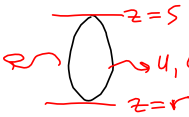


FIGURE 8 A type 3 region



front = $x = b$ $x = a$ = back = smaller x

$$\int_a^b \int_{z=z_1(x)}^{z=z_2(x)} \int_{y=u_1(x,z)}^{y=u_2(x,z)} dy\, dz\, dx$$

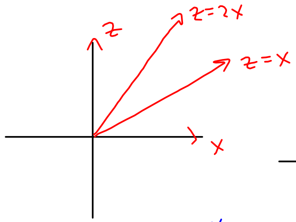


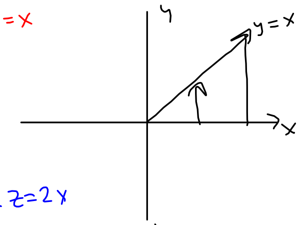
Farthest from yz-plane = Front = $x = u_2(z)$ $u_1(z)$ = Back (Closest to yz-plane

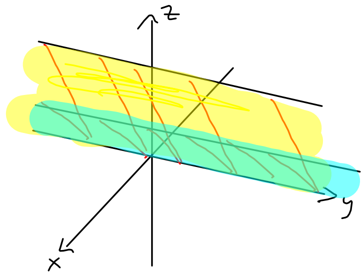
$$\int_r^s \int_{u_1(z)}^{u_2(z)} \int_{v_1(x,z)}^{v_2(x,z)} dy\, dx\, dz$$

9–18 Evaluate the triple integral.

10. $\iiint_E yz\cos(x^5)\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x,\ x \leqslant z \leqslant 2x\}$







$$\int_0^1\int_0^x\int_{z=x}^{z=2x} yz\cos(x^5)\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x \Big[yz^2\cos(x^5)\Big]_{x=z}^{2x=z} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x \Big(y[2x]^2\cos(x^5) - y[x]^2\cos(x^5)\Big)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x \Big(4x^2y\cos(x^5) - x^2y\cos(x^5)\Big)\,dy\,dx$$

$$= \frac{3}{2}\int_0^1\int_0^x x^2y\cos(x^5)\,dy\,dx = \frac{3}{4}\int_0^1 \Big[x^2y^2\cos(x^5)\Big]_{y=0}^{y=x} dx = \frac{3}{4\cdot 5}\int_0^1 5x^4\cos(x^5)\,dx$$

$$= \frac{3}{20}\Big[\sin(x^5)\Big]_0^1 = \boxed{\frac{3}{20}\sin(1)}$$

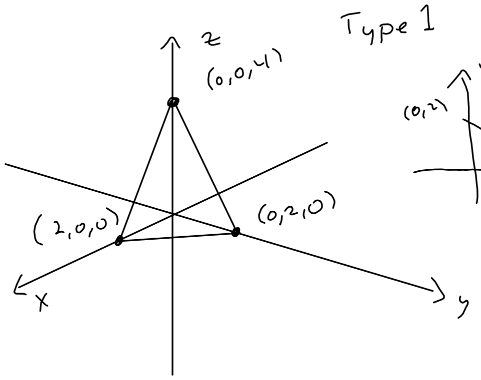
in radians

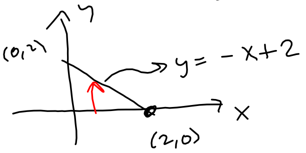
$u = x^5$
$du = 5x^4\,dx$

12. $\iiint_E y\,dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

(0, 0, 4)
(0, 2, 0)
(2, 0, 0)

Type 1





$$\int_0^2 \int_{0=y}^{y=-x+2} \int_{z=0}^{z=4-2x-2y} y \; dz\,dy\,dx$$

$$= \int_0^2 \int_{y=0}^{y=-x+2} \Big[yz \Big]_{z=0}^{z=4-2x-2y} dy\,dx = \int_0^2 \int_{y=0}^{y=-x+2} y\,[4-2x-2y]\,dy\,dx$$

$$= \int_0^2 \int_0^{y=2-x} (4y - 2xy - 2y^2)\,dy\,dx = \int_0^2 \left[2y^2 - xy^2 - \frac{2y^3}{3} \right]_0^{y=2-x} dx$$

$$= \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2(2-x)^3}{3} \right] dx = \int_0^2 \Big(2x^2 - 8x + 8 - x^3 + 4x^2 + 4x$$

$$- \frac{2}{3}\left[x^3 - 6x^2 + 12x - 8\right] \Big) dx = \int_0^2 \left(-x^3 + 6x^2 - 4x + 8 - \frac{2}{3}x^3 - 4x^2 - 8x + \frac{16}{3} \right) dx$$

Scratch

$$(x-2)^3 = x^3 - 3\cdot 2x^2 + 3\cdot 4x - 2^3 = x^3 - 6x^2 + 12x - 8$$

$$(2-x)^2 = (x-2)^2 = x^2 - 4x + 4$$

$$-\frac{2}{3}\left[x^3 - 6x^2 + 12x - 8\right] = -\frac{2}{3}x^3 + 4x^2 - 8x + \frac{16}{3}$$

$$= \int_0^1 \left(-\frac{5}{3}x^3 + 2x^2 - 12x + \frac{40}{3} \right) dx$$

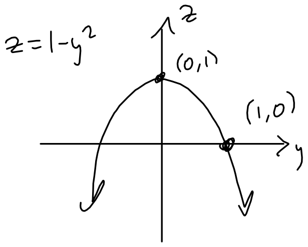
$$= \left[-\frac{5}{12}x^4 + \frac{2}{3}x^3 - 6x^2 + \frac{40}{3}x \right]_0^1 = -\frac{5}{12} + \frac{2}{3} - 6 + \frac{40}{3}$$

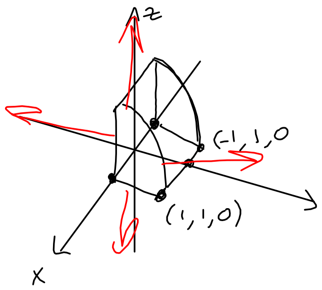
$$= \frac{-5 + 8 - 72 + 160}{12} = \frac{-77 + 168}{12} = \boxed{\frac{91}{12}}$$

$$\int_0^2 \int_0^{2-x} \int_0^{4-2\cdot x-2\cdot y} y\,dz\,dy\,dx \; \boxed{= \frac{4}{3}}$$

I messed up, SOMEwhere. Can you find it?

13. $\iiint_E x^2 e^y \, dV$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$





TI over TII

$$\int_0^1 \int_{-1}^1 \int_0^{1-y^2} dz\,dx\,dy$$

TI over TI

$$\int_{-1}^1 \int_0^1 \int_0^{1-y^2} dz\,dy\,dx$$

$\int y^2 e^y\,dy$ is integrate by parts, TWICE.
or just look up y^2e^y in formulas.

$$\iiint_E x^2 e^y\,dV = \int_{-1}^1 \int_0^1 \int_0^{1-y^2} x^2 e^y\,dz\,dy\,dx = \int_{-1}^1 \int_0^1 \Big[z\,x^2 e^y\Big]_0^{1-y^2} dy\,dx$$

$$= \int_{-1}^1 x^2 \int_0^1 [1-y^2]e^y\,dy\,dx = \int_{-1}^1 x^2 \int_0^1 (e^y - y^2 e^y)\,dy\,dx = \int_{-1}^1 x^2 \Big[e^y - (y^2 - 2y + 2)e^y\Big]_0^1 dx$$

$$= \int_{-1}^1 x^2 \Big[\big(e^1 - (1 - 2(1) + 2)e^1\big) - \big(e^0 - (0^2 - 2(0) + 2)e^0\big)\Big]\,dx$$

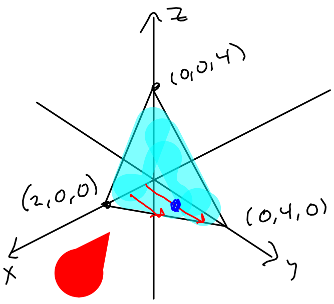
$$= \int_{-1}^1 x^2 \Big[(e - 1e^1) - (1-2)\Big]dx = \int_{-1}^1 x^2\,dx = \frac{1}{3}x^3\Big]_{-1}^1 = \frac{1}{3}(1) - \left(\frac{1}{3}\right)(-1)$$

$$\boxed{= \frac{2}{3}}$$

15. $\iiint_T x^2\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$



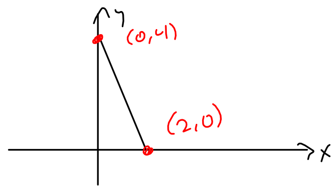
Intercepts: $(0,0,4), (0,4,0), (2,0,0)$

$2x+y+z=4 \Rightarrow$

$z = 4-2x-y$

Type I over Type I

$$\int_0^2\int_0^{-2x+4}\int_0^{4-2x-y} dz\,dy\,dx$$



$m = \frac{4-0}{0-2} = -2$

$y = -2(x-0)+4$

$y = -2x+4$

$$\int_0^2\int_0^{-2x+4}\int_0^{4-2x-y} dz\,dy\,dx = \int_0^2\int_0^{-2x+4}\left[z\right]_0^{4-2x-y} dy\,dx = \int_0^2\int_0^{-2x+4}(4-2x-y)\,dy\,dx$$

$$= \int_0^2\left[(4-2x)y - \frac{y^2}{2}\right]_0^{-2x+4} dx = \int_0^2\left[\underline{(4-2x)(-2x+4)} - \underline{\frac{(-2x+4)^2}{2}}\right]dx$$

$$= \int_0^2 \frac{4x^2-16x+16}{2}dx = \int_0^2(2x^2-8x+8)dx = \left[\frac{2}{3}x^3 - 4x^2 + 8x\right]_0^2$$

$$= \frac{16}{3} - 16 + 16 = \boxed{\frac{16}{3}}$$

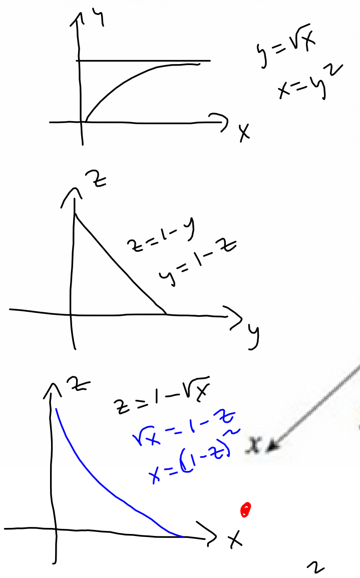
$(4-2x)(-2x+4) = (2x-4)^2$

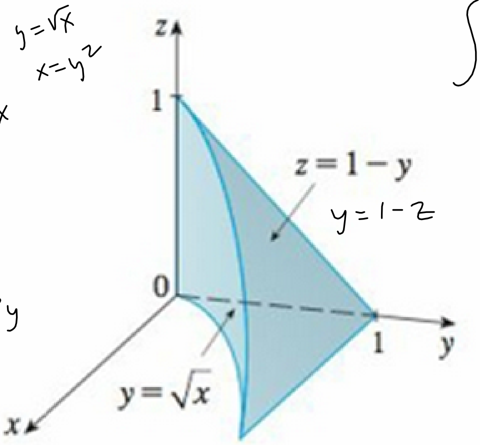
$(a-b)^2 = (b-a)^2 = a^2-2ab+b^2$

33. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$ T1 over TI

Rewrite this integral as an equivalent iterated integral in the five other orders.





$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\, dy\, dx\, dz$$

$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f\, dy\, dz\, dx$$

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} f\, dx\, dy\, dz$$

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f\, dx\, dz\, dy$$

$$\int_0^1 \int_0^{y^2} \int_0^{1-y} f\, dz\, dx\, dy$$

34. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.