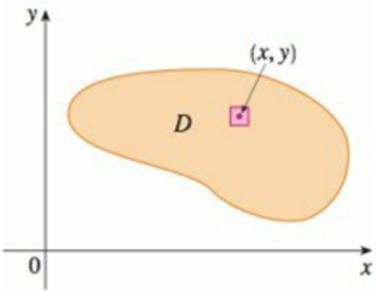
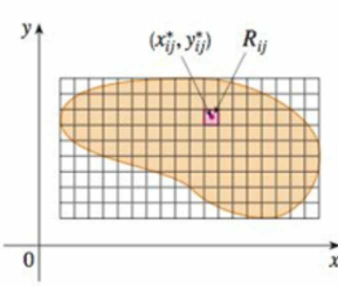
16.5 Applications of Double Integrals





Mass Density at $(x, y) = \rho(x, y) = \lim \dfrac{\Delta m}{\Delta A}$

Mass of the ij^{th} rectangle $\approx \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$

$$m \approx \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A \qquad \iint_D \rho(x, y)\,dA = m$$

Total Charge from Charge Density: $Q = \iint_D \sigma(x, y)\,dA$

Moments and Center of Mass

Recall that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis.

$$[\rho(x_{ij}^*, y_{ij}^*)\,\Delta A]\,y_{ij}^*$$

Prior to this, we assumed the density function was constant. Now, we do not.

moment about the x-axis:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D y\,\rho(x,y)\,dA$$

moment about the y-axis

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D x\,\rho(x,y)\,dA$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

where $m = \iint_D \rho(x,y)\,dA$

Moment of Inertia about an Axis:

moment of inertia (also called the **second moment**) of a particle of mass m about an axis is: mr^2, where r is the distance from the particle to the axis.

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

POLAR Moment of Inertia is the moment about the origin:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2) \rho(x, y)\, dA$$

Probability Density Function f of a continuous random variable x has the properties

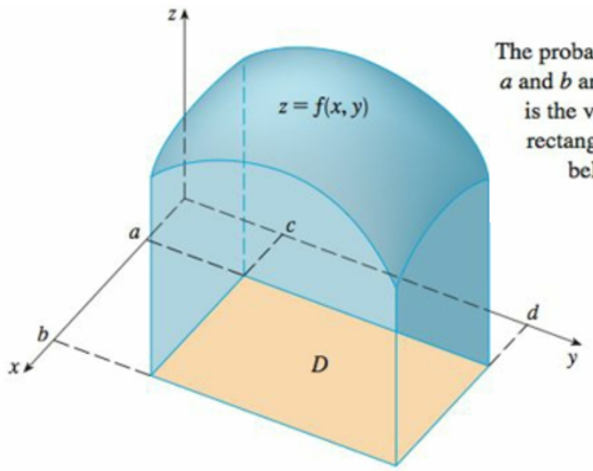
$f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Now we have **joint probability density function:**

$$P((X, Y) \in D) = \iint_D f(x, y)\,dA$$

$$P(a \leq X \leq b,\ c \leq Y \leq d) = \int_a^b \int_c^d f(x, y)\,dy\,dx$$



The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.

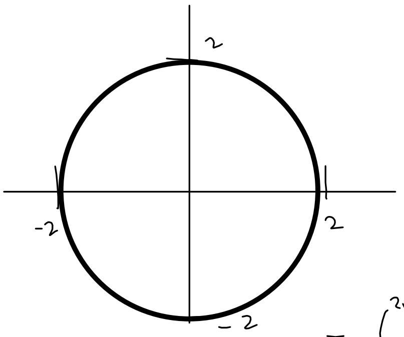
$$\iint_{\mathbb{R}^2} f(x, y)\,dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$$

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

1. Electric charge is distributed over the rectangle $1 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 2$ so that the charge density at (x, y) is $\sigma(x, y) = 2xy + y^2$ (measured in coulombs per square meter). Find the total charge on the rectangle.

2. Electric charge is distributed over the disk $x^2 + y^2 \leqslant 4$ so that the charge density at (x, y) is $\sigma(x, y) = x + y + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

I'll do #2.



$$x^2+y^2 \le 4$$
$$r^2 \le 4$$
$$r \le 2$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{r=2} (x+y+x^2+y^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(r\cos\theta + r\sin\theta + r^2\cos^2\theta + r^2\sin^2\theta\right) r\, dr\, d\theta$$

(Scratch: $r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$)

$$= \int_0^{2\pi}\int_0^2 \left(r\cos\theta + r\sin\theta + r^2\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \left(r^2\cos\theta + r^2\sin\theta + r^3\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{3}r^3(\cos\theta+\sin\theta) + \tfrac{1}{4}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{8}{3}(\cos\theta+\sin\theta) + 4\right) d\theta = \underbrace{\tfrac{8}{3}\Big[\sin\theta + \cos\theta\Big]_0^{2\pi}}_{\text{Goes Away}} + \tfrac{32}{3}\Big[\theta\Big]_0^{2\pi} + 4\theta\Big]_0^{2\pi}$$

$$\sin 2\pi - \cos 2\pi - [\sin 0 - \cos 0]$$
$$-1 - [-1] = 0$$

$$= \boxed{8\pi\ C}$$

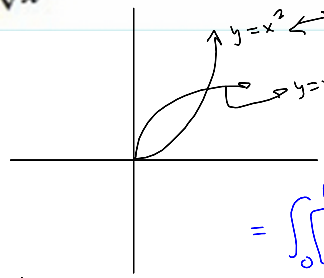
'C' for Coulombs.

3–10 Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

5. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

10. D is bounded by the parabolas $y = x^2$ and $x = y^2$; $\rho(x, y) = \sqrt{x}$

$\sqrt{x} = x^{\frac{1}{2}} \rightsquigarrow \frac{2}{3}x^{\frac{3}{2}}$



TII

$$\text{MASS} = \int_0^1 \int_{y^2=x}^{x=\sqrt{y}} \sqrt{x}\, dx\, dy$$

$$= \int_0^1 \left[\frac{2}{3}x^{\frac{3}{2}}\right]_{y^2=x}^{\sqrt{y}=x=y^{\frac{1}{2}}} dy = \int_0^1 \frac{2}{3}\left[(y^{\frac{1}{2}})^{\frac{3}{2}} - (y^2)^{\frac{3}{2}}\right] dy$$

$$= \frac{2}{3}\int_0^1 \left(y^{\frac{3}{4}} - y^3\right) dy = \frac{2}{3}\left[\frac{4}{7}y^{\frac{7}{4}} - \frac{1}{4}y^4\right]_0^1 = \frac{2}{3}\left[\frac{4}{7} - \frac{1}{4}\right]$$

$$= \frac{2}{3}\left[\frac{16-7}{28}\right] = \frac{2}{3}\left[\frac{9}{28}\right] = \boxed{\frac{3}{14} \text{ mass unit}} = m$$

$$M_y = \iint x\rho\, dA \qquad M_x = \iint y\rho\, dA$$

ORDER

$$\boxed{\text{TI} \quad \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} dy\, dx} \Longrightarrow M_y = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} x\sqrt{x}\, dy\, dx = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} x^{\frac{3}{2}}\, dy\, dx$$

$$= \int_0^1 \left[y\, x^{\frac{3}{2}}\right]_{y=x^2}^{y=\sqrt{x}=x^{\frac{1}{2}}} dx = \int_0^1 \left(x^{\frac{1}{2}}(x^{\frac{3}{2}}) - x^2(x^{\frac{3}{2}})\right) dx = \int_0^1 \left(x^2 - x^{\frac{7}{2}}\right) dx$$

$$= \left[\frac{1}{3}x^3 - \frac{2}{9}x^{\frac{9}{2}}\right]_0^1 = \frac{3}{9} - \frac{2}{9} = \boxed{\frac{1}{9} = M_y}$$

$$M_x = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} y\, x^{\frac{1}{2}}\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[y^2 x^{\frac{1}{2}}\right]_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 \left((\sqrt{x})^2 x^{\frac{1}{2}} - (x^2)^2 x^{\frac{1}{2}}\right) dx = \frac{1}{2}\int_0^1 \left(x^{\frac{3}{2}} - x^{\frac{9}{2}}\right) dx$$

$$= \frac{1}{2}\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{11}x^{\frac{11}{2}}\right]_0^1 = \frac{1}{2}\left[\frac{2}{5} - \frac{2}{11}\right] = \frac{12}{55}\cdot\frac{1}{2} = \frac{6}{55}$$

Summarize: $m = \frac{3}{14}$, $M_y = \frac{1}{9}$, $M_x = \frac{6}{55}$

Center of mass $= (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \frac{14}{3}\left(\frac{1}{9}, \frac{6}{55}\right)$

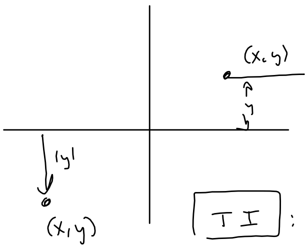
$$= \left(\frac{14}{3}\cdot\frac{1}{9}, \frac{14}{3}\cdot\frac{6}{55}\right) = \boxed{\left(\frac{14}{27}, \frac{28}{55}\right) = (\bar{x}, \bar{y})}$$

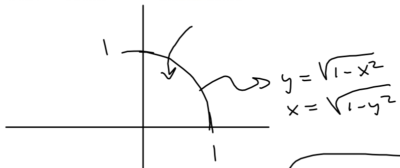
11. A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

12. Find the center of mass of the lamina in Exercise 11 if the density at any point is proportional to the square of its distance from the origin.

y proportional to x means $y = kx$ for some $k \in \mathbb{R}$.

Distance from x-axis for (x,y) is $|y|$





$x^2+y^2=1$

So $\rho(x,y) = k|y|$ for some $k \in \mathbb{R}$

QI means $y \geq 0 \Rightarrow$ $\boxed{\rho(x,y) = ky}$

$\boxed{\text{TI}}$: $\iint dy\,dx$

$$m = \iint \rho(x,y)\,dA$$

$$\text{TI}: \int_{x=0}^{x=1}\int_{y=0}^{\sqrt{1-x^2}=y} ky\,dy\,dx = k\int_0^1 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{k}{2}\int_0^1 \left(\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{k}{2}\int_0^1 (1-x^2)\,dx = \frac{k}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{k}{2}\cdot\left[1-\frac{1}{3}\right] = \frac{k}{2}\cdot\frac{2}{3} = \boxed{\frac{k}{3} = m}$$

$$M_y = \iint x\rho(x,y)\,dA \overset{\boxed{\text{TI}}}{=} k\int_0^1 x\int_0^{\sqrt{1-x^2}} y\,dy\,dx = \frac{k}{2}\int_0^1 x\left[y^2\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{k}{2}\int_0^1 x(1-x^2)\,dx = \frac{k}{2}\int_0^1 (x-x^3)\,dx = \frac{k}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{k}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \boxed{\frac{k}{8} = M_y}$$

M_x: Exploit Circular Region:

$$M_x = \iint y\rho(x,y)\,dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{1} y\cdot ky\,dy\,dx$$

$$= k\int_0^{\frac{\pi}{2}}\int_0^1 r^2\sin^2\theta\, r\,dr\,d\theta = k\int_0^{\frac{\pi}{2}}\int_0^1 r^3\sin^2\theta\,dr\,d\theta$$

$$= k\int_0^{\frac{\pi}{2}} \sin^2\theta\left[\frac{r^4}{4}\right]_0^1 d\theta = k\int_0^{\frac{\pi}{2}} \sin^2\theta\cdot\frac{1}{4}\,d\theta = \frac{k}{4}\int_0^{\frac{\pi}{2}}\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)d\theta$$

$$= \frac{k}{4}\left[\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} - \frac{k}{8}\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos(2\theta)\cdot 2\,d\theta$$

$u = 2\theta$
$du = 2\,d\theta$

$$= \frac{k}{4}\left[\frac{\pi}{4}\right] - \underbrace{\frac{k}{16}\sin(2\theta)\Big]_0^{\frac{\pi}{2}}}_{0} = \boxed{\frac{k\pi}{16} = M_x}$$

$$\Rightarrow (\bar{x},\bar{y}) = \left(\frac{3}{k}\left(\frac{k}{8}\right), \frac{3}{k}\left(\frac{k\pi}{16}\right)\right)$$

$$= \boxed{\left(\frac{3}{8}, \frac{3\pi}{16}\right) = \text{center of mass}}$$

28. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.

(b) If X and Y are random variables whose joint density function is the function f in part (a), find

(i) $P\left(X \geqslant \frac{1}{2}\right)$ (ii) $P\left(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}\right)$

(c) Find the expected values of X and Y.