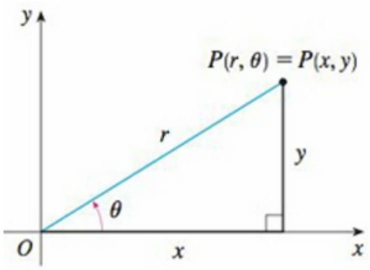
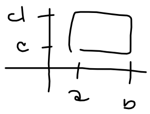
15.4 Double Integrals in Polar Coordinates.



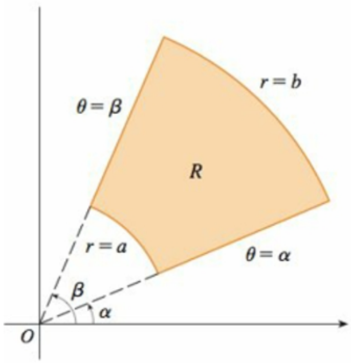
Recall

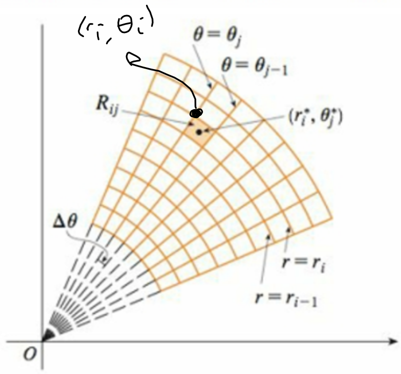
$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$



"Rectangular" Rectangle: $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\} = [a,b] \times [c,d]$

polar rectangle $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

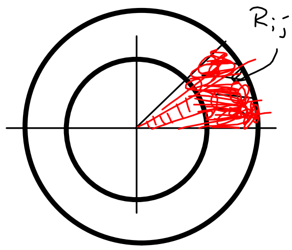




The "center" of the polar rectangle $R_{ij} = \{(r, \theta) \mid r_{i-1} \leqslant r \leqslant r_i, \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$

has coordinates $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ $\qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

Recall: The area of a sector of a circle $A = \frac{1}{2}r^2\theta$



r
θ

Area of $R_{ij} = \Delta A_{ij} = \frac{1}{2}r_i^2\,\Delta\theta - \frac{1}{2}r_{i-1}^2\,\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta$

$$= \underbrace{\tfrac{1}{2}(r_i + r_{i-1})}_{r_i^*}(r_i - r_{i-1})\,\Delta\theta = r_i^*\,\Delta r\,\Delta\theta$$

$\Delta r, \Delta\theta \to 0 \to r\,dr\,d\theta$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

Define $g(r, \theta) = r f(r\cos\theta, r\sin\theta)$. Then the above Riemann Sum becomes:

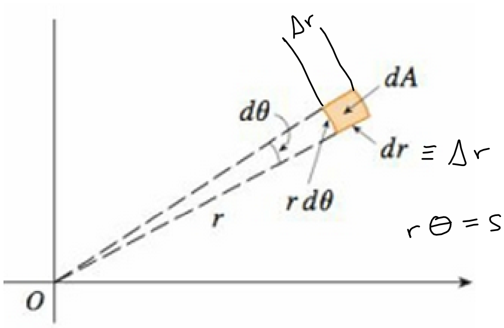
$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

One hopes and expects that in the limit as m, n approach infinity, we obtain a double integral:

2 CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

A nice way of remembering the area increment dA in polar coordinates is by thinking of the polar rectangle as an ordinary rectangle:



$r\theta = s =$ arc length

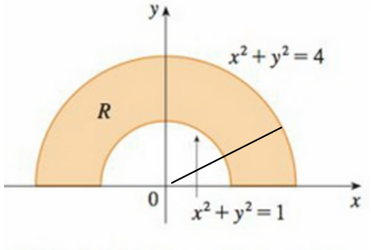
And the smaller we get, the more the arcs involved ARE like straight line segments and the more "parallel" the sides are.

Personally, I just think it's a joke. Hardee-Har! ("r dr")

Why fool with this?

Because some regions are more circular than others, and polar coordinates give us an efficient way to represent things in some cases.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



(b) $R=\{(r,\theta)\mid 1\le r\le 2,\ 0\le\theta\le\pi\}$

R is a polar rectangle.

R is a polar rectangle, so...

$$\int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 3r\cos\theta\, r\,dr\,d\theta + 4\int_0^{\pi}\int_1^2 r^2\left(\tfrac{1}{2}(1-\cos(2\theta))\right) r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 3r^2\cos\theta\,dr\,d\theta + \frac{4}{2}\int_0^{\pi}\int_1^2 r^3\,(1-\cos(2\theta))\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\Big[r^3\cos\theta\Big]_{r=1}^{r=2}d\theta + 2\int_0^{\pi}\left[\frac{r^4}{4}(1-\cos(2\theta))\right]_{r=1}^{2}d\theta$$

$$= \int_0^{\pi}(2^3-1^3)\cos\theta\,d\theta + \frac{1}{2}\int_0^{\pi}(2^4-1^4)(1-\cos(2\theta))\,d\theta$$

$$= \int_0^{\pi}7\cos\theta\,d\theta + \frac{15}{2}\int_0^{\pi}(1-\cos(2\theta))\,d\theta$$

Power-reducing formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= \Big[7\sin\theta\Big]_0^{\pi} + \frac{15}{2}\left[\int_0^{\pi}d\theta - \frac{1}{2}\int_0^{\pi}2\cos(2\theta)\,d\theta\right] = 7(\sin(\pi)-\sin(0)) + \left[\frac{15}{2}\theta\right]_0^{\pi} - \frac{1}{2}\Big[(-\sin(2\theta))\Big]_0^{\pi}$$

$$= 0 + \frac{15\pi}{2} + \frac{1}{2}\Big[\sin(2\pi)-\sin(0)\Big] = \boxed{\frac{15\pi}{2}}$$

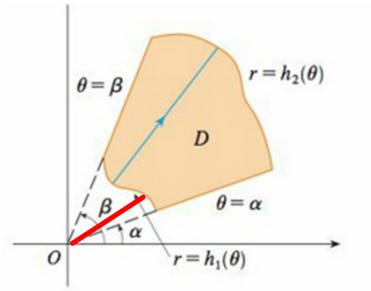


FIGURE 7
$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$

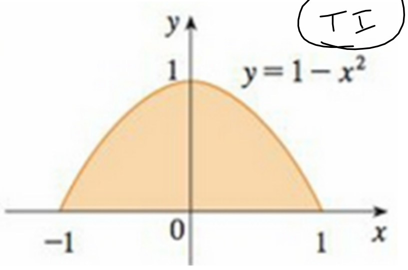
3 If f is continuous on a polar region of the form

$$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$$

then $$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$$

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

2.

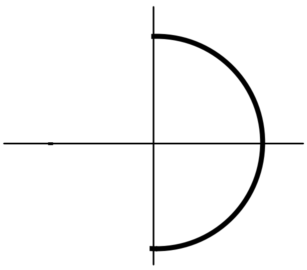


TI

$$\int_{-1}^{1} \int_{y=0}^{y=1-x^2} f(x,y)\, dy\, dx$$

7–14 Evaluate the given integral by changing to polar coordinates.

11. $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis



$x^2 = 4-y^2 \Longrightarrow x^2+y^2=4$

$e^{-r^2\cos^2\theta - r^2\sin^2\theta}$

$= e^{-r^2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}\,r\,dr\,d\theta$$

$$-\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}(-2r\,dr)\,d\theta = -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[e^{-r^2}\right]_0^2 d\theta$$

$$= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[e^{-4}-e^{-0}\right]dy = -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(e^{-4}-1)\,d\theta$$

$$= -\frac{1}{2}\theta\left(e^{-4}-1\right)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[-\frac{1}{2}(2)\,\theta\left(e^{-4}-1\right)\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2}\left(e^{-4}-1\right) = \frac{\pi}{2}-\frac{\pi}{2e^4} = \frac{e^4\pi-\pi}{2e^4}$$

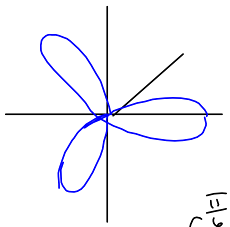
15–18 Use a double integral to find the area of the region.

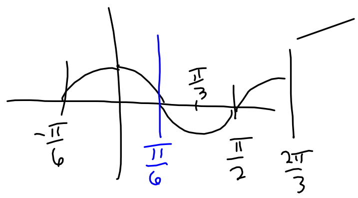
15. One loop of the rose $r = \cos 3\theta$

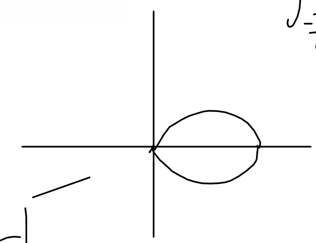
$f(x,y) = 1 = f(r,\theta)$

$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$

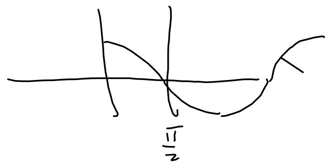
$\cos(3\theta)$







$\cos\theta$



$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{\cos(3\theta)} 1\, r\,dr\,d\theta = 2\int_0^{\frac{\pi}{6}}\int_0^{\cos(3\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}}\left[\frac{1}{2}r^2\right]_0^{\cos(3\theta)} d\theta = \int_0^{\frac{\pi}{6}}\cos^2(3\theta)\,d\theta$$

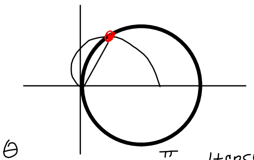
$$\frac{1}{2}\int_0^{\frac{\pi}{6}}(1+\cos(6\theta))\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{6}} d\theta + \frac{1}{6}\int_0^{\frac{\pi}{6}}\cos(6\theta)\,6\,d\theta\right] = \frac{1}{2}\left[\left[\theta\right]_0^{\frac{\pi}{6}} - \frac{1}{6}\left[\sin(6\theta)\right]_0^{\frac{\pi}{6}}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0 - \frac{1}{6}(\sin\pi - \sin 0)\right] = \boxed{\frac{\pi}{12}}$$

15–18 Use a double integral to find the area of the region.

18. The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$



$3\cos\theta = 1 + \cos\theta \Longrightarrow$

$\cos\theta = \frac{1}{2} \rightarrow$

$\theta = \frac{\pi}{3}$ here

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\frac{\pi}{2}}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta = A + B$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[r^2\right]_{3\cos\theta}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\cos^2\theta + 2\cos\theta + 1 - 9\cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(-8\cos^2\theta + 2\cos\theta + 1\right) d\theta = \frac{1}{2}\int -8\cos^2\theta\, d\theta + \frac{1}{2}\int \left(2\cos\theta + 1\right) d\theta$$

$$= C + D$$

$$C = -4\int \frac{1}{2}\left(1 + \cos(2\theta)\right) d\theta$$

$$= -4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\, d\theta - 2\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos(2\theta)\cdot 2\, d\theta = -2\left[\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\sin(2\theta)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -\frac{2\pi}{2} - \left(-\frac{2\pi}{3}\right) - \left[\sin\pi - \sin\left(\frac{2\pi}{3}\right)\right] = \boxed{-\frac{\pi}{3} + \frac{\sqrt{3}}{2} = C}$$

$$D = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(2\cos\theta + 1\right) d\theta = \frac{1}{2}\left[2\sin\theta + \theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{2}\left[2 + \frac{\pi}{2} - \left(\sqrt{3} + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[2 + \frac{\pi}{2} - \sqrt{3} - \frac{\pi}{3}\right] = \frac{1}{2}\left[2 - \sqrt{3} + \frac{\pi}{6}\right] = \boxed{1 - \frac{\sqrt{3}}{2} + \frac{\pi}{3} = D}$$

$$B = \int_{\frac{\pi}{2}}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left[r^2\right]_0^{\cos\theta + 1} d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(\cos^2\theta + 2\cos\theta + 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{2} + 2\cos\theta\right) d\theta + \frac{1}{2}\cdot\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \cos(2\theta)\cdot 2\, d\theta$$

(Scratch: $\cos^2\theta + 2\cos\theta + 1 = \frac{1}{2}\left(1 + \cos 2\theta\right) + 2\cos\theta + 1$

$= \frac{1}{2} + \cos 2\theta + 2\cos\theta + 1 = \frac{3}{2} + \cos 2\theta + 2\cos\theta$)

$$= \frac{1}{2}\left[\frac{1}{2}\theta + 2\sin\theta\right]_{\frac{\pi}{2}}^{\pi} + \left[\frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{2}\left[\frac{\pi}{2} + 2\cdot 0 - \left(\frac{\pi}{4} + 2\right)\right]$$

$$+ \frac{1}{4}\left[0\right] = \frac{\pi}{4} - \frac{\pi}{8} + 1 = \boxed{\frac{\pi}{8} + 1 = B}$$

& so Area $= A + B = C + D + B = -\frac{\pi}{3} + \frac{\sqrt{3}}{2} + 1 - \frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{\pi}{8} + 1$

$$= \boxed{\cancel{\frac{\sqrt{3}}{2}} + 2 + \frac{\pi}{8} = \frac{\text{Area}}{2}}$$

Tech Assist: $2\cdot\left(\int_{\frac{Pi}{3}}^{\frac{Pi}{2}}\int_{3\cdot\cos(x)}^{1+\cos(x)} r\,dr\,dx + \int_{\frac{Pi}{2}}^{Pi}\int_{0}^{\cos(x)+1} r\,dr\,dx\right) = \frac{\pi}{4}$

36. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

Pretty cool idea that my mentor used to ease me into Complex Analysis and Residue Theory.