Section 15.1 Double Integrals over Rectangles.

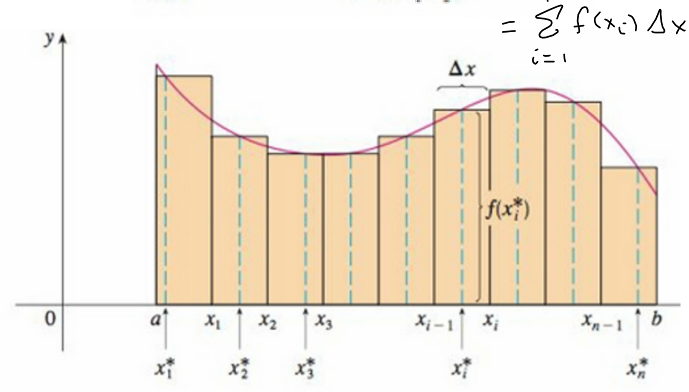
S 15.1 from the 16.1 handout,

#s 1, 5, 11, 14

Recall $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

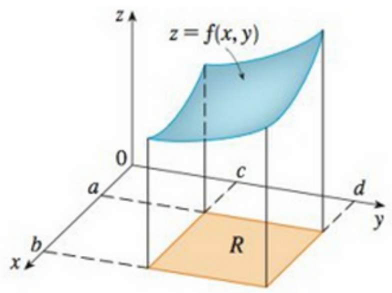
$= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$

$= \sum_{i=1}^{n} f(a + i\Delta x)\,\Delta x$

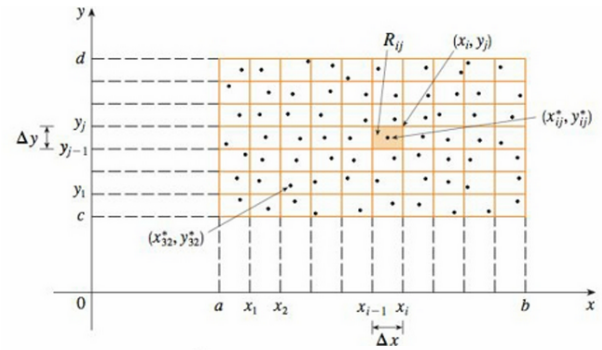


$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$

Suppose $f(x, y) \ge 0$. Want to find the volume under the surface.



$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

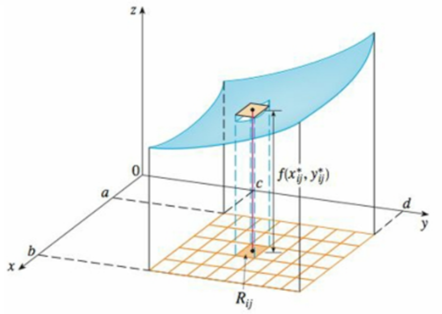


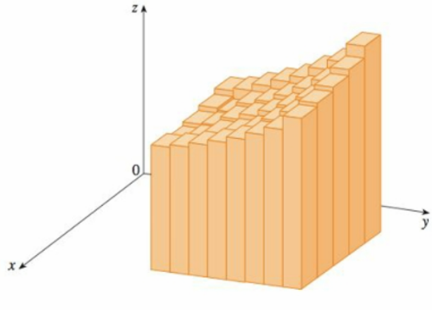
Volume of the (i, j)th column is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \rightarrow \Delta y \Delta x \text{ or } \Delta x \Delta y$$

Process: $i = 1$. Run through the sum of the j's.

$i = 2$. Repeat....





$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$ gives the exact volume. This is calculus, so...

The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\,dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

if this limit exists. → Continuity is sufficient.

The last double sum is obtained by picking the top right corner of each rectangle. This makes no difference in the result, but makes it easier to write the thing out (and calculate it).

1. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq 6, 0 \leq y \leq 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square.

Midpoint Rule is all well and good, and you might use it one day to do a digital estimate of a volume or double integral.

But for our purposes, upper-right corner rule is the 2-D version of right-endpoint Riemann sums of the 1-D case.

$m = 3,\ n = 2$

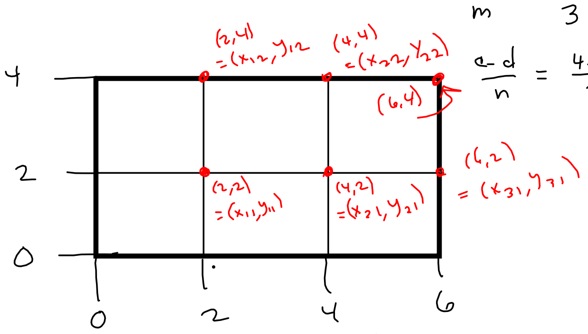
x_{11}, x_{12}

x_{21}, x_{22}

x_{31}, x_{32}

$$\frac{b-a}{m} = \frac{6-0}{3} = 2 = \Delta x$$

$$\frac{c-d}{n} = \frac{4-0}{2} = 2 = \Delta y$$



$$\Delta A = \Delta x \Delta y = 2 \cdot 2 = 4$$

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_{ij}, y_{ij})\,\Delta A =$$

$$= f(x_{11}, y_{11}) \cdot 4 + f(x_{12}, y_{12}) \cdot 4 + f(x_{21}, y_{21}) \cdot 4 + f(x_{22}, y_{22}) \cdot 4$$

$$+ f(x_{31}, y_{31}) \cdot 4 + f(x_{32}, y_{32}) \cdot 4$$

$$= 4\left[f(2,2) + f(2,4) + f(4,2) + f(4,4) + f(6,2) + f(6,4) \right]$$

$$= 4\left[(2)(2) + (2)(4) + (4)(2) + (4)(4) + (6)(2) + (6)(4) \right]$$

$$= 4\left[4 + 8 + 8 + 16 + 12 + 24\right] = 4\left[72\right] = \boxed{288}$$

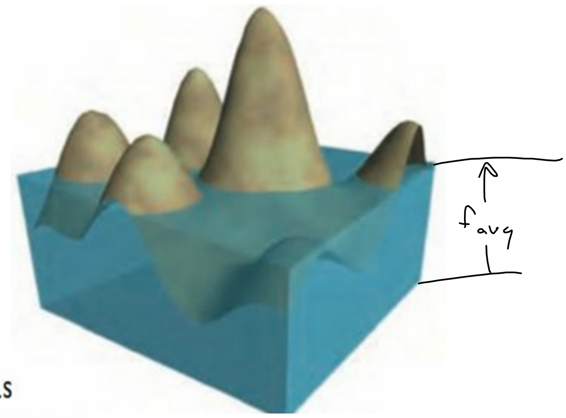
Recall AVERAGE VALUE $= f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

We bump it up one dimension:

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

One consequence of this is obtained by multiplying both sides of this equation by the Area under the surface:

$$A(R)\,f_{\text{ave}} = \iint_R f(x,y)\,dA$$



PROPERTIES OF DOUBLE INTEGRALS

7 $$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

8 $$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \quad \text{where } c \text{ is a constant}$$

If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

9 $$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

Integral Operator is a Linear Operator.

$$\iint (cf + dg) = c\iint f + d\iint g$$

f, g funcs.

c, d real #s.