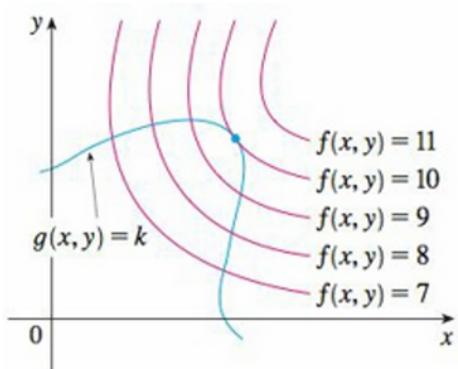


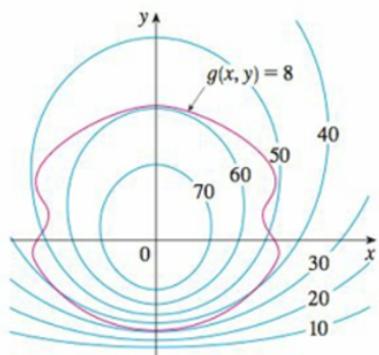
Section 14.8 Lagrange Multipliers

**FIGURE 1**

Trying to maximize or minimize f , subject to some sort of constraint g .

The big observation to make is that the tangent lines are the same for the level curves and the curve corresponding to $g = k$.

- I. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



3-17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 + y^2; \quad xy = 1$

3-17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

$$6. f(x, y) = e^{xy}; \quad x^3 + y^3 = 16, \quad g(x, y) = x^3 + y^3$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle ye^{xy}, xe^{xy} \right\rangle$$

$$\nabla g = \left\langle 3x^2, 3y^2 \right\rangle \quad \text{s.t. } g(x, y) = 16$$

$$\nabla f = \lambda \nabla g \implies \lambda = \frac{ye^{xy}}{3x^2}, \quad xe^{xy} = 3\lambda y^2 \quad * \quad * \quad x=0 \text{ forces } y=0, b/c$$

$$\lambda = \frac{ye^{xy}}{3x^2}, \quad \lambda = \frac{xe^{xy}}{3y^2} \quad \text{if } e^{xy} \neq 0.$$

$$\lambda = \lambda \implies \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$$

$$\implies 3y^3e^{xy} = 3x^3e^{xy}$$

$$y^3 = x^3 \\ y = x \quad \text{in general.}$$

Now send that to $g(x, y) = 16$

$$x^3 + y^3 = x^3 + x^3 - 2x^3 = 16 \implies$$

$$x^3 = 8 \implies$$

$$x = 2 \implies y = 2$$

$$\implies f(2, 2) = e^{2(2)} = e^4$$

3-17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

10. $f(x, y, z) = x^2y^2z^2; \quad x^2 + y^2 + z^2 = 1 \Rightarrow g(x, y)$

$$\nabla f = \langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z \rangle$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

$$\nabla f = \lambda \nabla g \Rightarrow$$

$$2xy^2z^2 = 2\lambda x, \quad 2x^2yz^2 = 2\lambda y, \quad 2x^2y^2z = 2\lambda z$$

$$x = y^2z^2, \quad \lambda = x^2z^2$$

$$\lambda = x^2y^2$$

$$y^2z^2 = x^2z^2 \quad \text{3 times!}$$

$$y^2z^2 = x^2y^2$$

$$y = \pm z$$

$$x^2 = z^2$$

$$y^2 = x^2$$

$$z = \pm x$$

$$x^2 + y^2 + z^2 = 3x^2 = 1 \Rightarrow$$

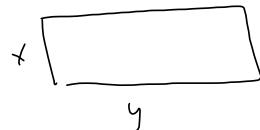
$$x = \pm \sqrt{\frac{1}{3}}$$

This gives $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),$

$$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$

$$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$

25. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter P is a square.



$$A = f(x, y) = xy$$

s.t. $g(x, y) = 2x + 2y = P$

$$\nabla f = \langle y, x \rangle$$
$$\lambda \nabla g = \lambda \langle 2, 2 \rangle \Rightarrow y = 2\lambda, x = 2\lambda$$
$$\Rightarrow x = y = 2\lambda$$

! /

27-39 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 15.7.

- 39.** Find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.

Let $(x, y, z) \in P$. Then distance to $(2, 1, -1)$ is

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2} \Rightarrow$$

$$\begin{aligned} f(x, y, z) &= (x-2)^2 + (y-1)^2 + (z+1)^2 \text{ to be minimized} \\ \text{s.t. } x+y-z &= g(x, y, z) = 1 \end{aligned}$$

$$\begin{aligned} \nabla f &= \langle 2(x-2), 2(y-1), 2(z+1) \rangle \\ &= \langle 2x-4, 2y-2, 2z+2 \rangle \end{aligned}$$

$$2 \nabla g = \lambda \langle 1, 1, -1 \rangle$$

$$\Rightarrow \lambda = 2x-4, \quad \lambda = 2y-2, \quad -\lambda = 2z+2$$

$$\begin{aligned} 2x &= \lambda + 4 & 2y &= \lambda + 2 & 2z &= -\lambda - 2 \\ x &= \frac{\lambda + 4}{2} & y &= \frac{\lambda + 2}{2} & z &= \frac{-\lambda - 2}{2} \end{aligned}$$

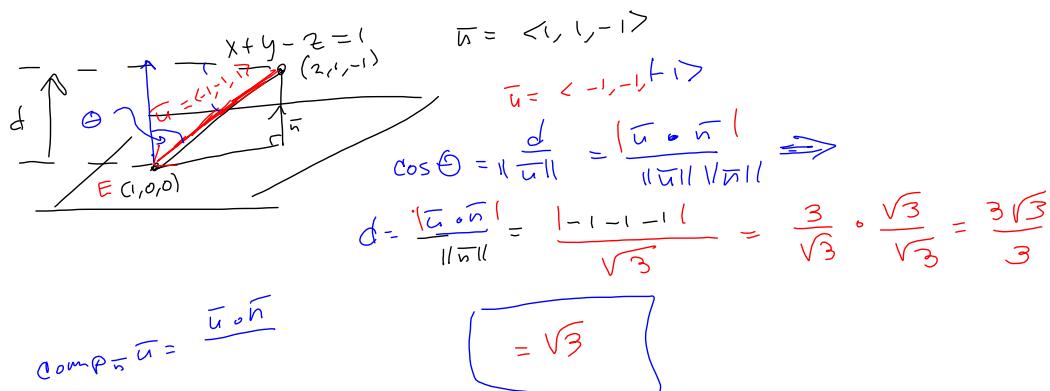
$$\frac{\lambda + 4}{2} + \frac{\lambda + 2}{2} - \frac{-\lambda - 2}{2} = 1 \quad x = \frac{-2+4}{2} = 1 = x$$

$$\lambda + 4 + \lambda + 2 + \lambda + 2 = 2 \quad y = \frac{-2+2}{2} = 0 = y$$

$$\begin{aligned} 3\lambda &= -6 \\ \lambda &= -2 \end{aligned}$$

$z = 0$
 $(1, 0, 0)$ is closest!

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2} = \boxed{\sqrt{3}}$$



45. (a) Find the maximum value of

$$f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that x_1, x_2, \dots, x_n are positive numbers and
 $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

- (b) Deduce from part (a) that if x_1, x_2, \dots, x_n are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?