$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$z = f(x, y)$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

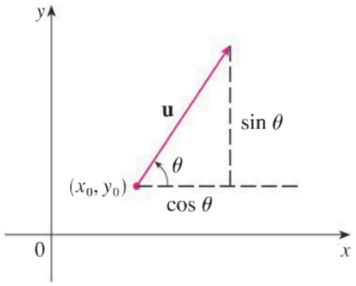


FIGURE 2
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$
$\mathbf{u} = \langle a, b \rangle = \langle \cos u, \sin u \rangle$

Not too sure what that last bit is...

14.6 Directional Derivatives and the Gradient Vector

Where we try to convince you that it *all* boils down to the gradient and that the partials with respect to x and y **entirely** describe the "tilt" of a surface at a point. First, we define **Directional Derivative**.

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

This is the "tilt" of the surface in the direction of **u**.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

Define $g(h) = f(x_0 + ha, y_0 + hb)$ by basically holding all the other variables fixed.

4 $$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives $f(x,y) = f(x(h), y(h))$

$\frac{dx}{dh} = a$

See Page 978 $$\frac{df}{dh} = g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\,a + f_y(x, y)\,b$$

$\frac{dy}{dh} = b$

$x = x(h)$, $y = y(h)$,

because everything else there is fixed.

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

5 $$g'(0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

4 $$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

This concludes the proof.

we're saying that $f_x\bar{i}$, $f_y\bar{j}$ spans the tangent plane, and can give you $D_{\bar{u}}f$, by writing $\bar{u} = a\bar{i} + b\bar{j}$ & $D_{\bar{u}}f = f_x$

When the angle that **u** makes with the positive x-axis is handy, and since **u** is of length 1, we obtain:

If the unit vector **u** makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

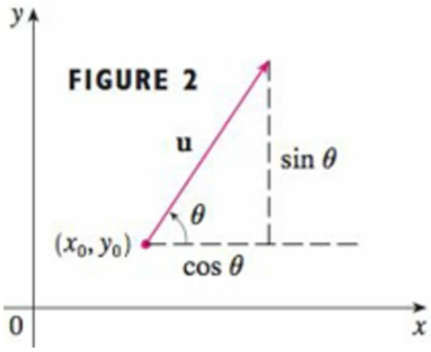


FIGURE 2

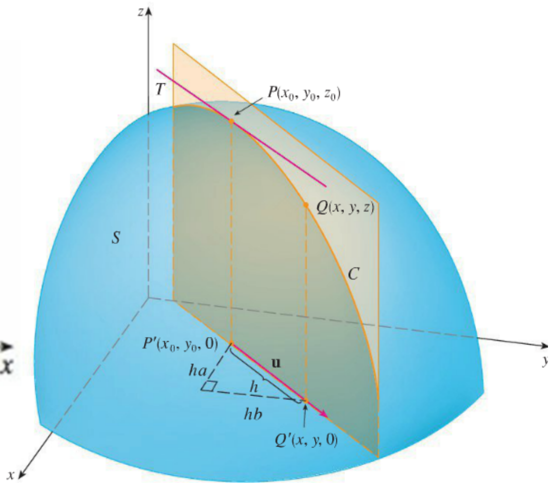
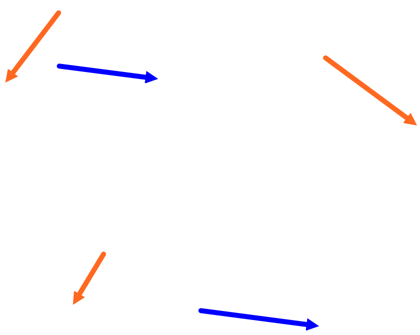


FIGURE 3



$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$ for some (nonzero) h.

$x - x_0 = ha,\ y - y_0 = hb$, so $x = x_0 + ha,\ y = y_0 + hb$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Gradient Vector: NOTATION !

Notice that you could write the directional derivative as a dot product?

$\bar{u} = \langle a, b \rangle = \langle u_1, u_2 \rangle$

$D_{\bar{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$