Section 14.5 The Chain Rule

14.5 #s 1, 4, 7, 10, 13, (17-20 (optional)), 24, 27, 32, 35, 43*, 45*

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Note the interplay of $d - vs - \partial$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Note that we're no longer talking about a space curve, but a surface, now. When x and y both depended on one (the same) parameter t, the function was a 1-dimensional object embedded in 3-space. When they each depend on 2 parameters, you get a 2-dimensional object embedded in 3-space (a surface).

In the 2-parameter case, we say that s and t are **independent** variables,

x and y are **intermediate** variables, and z is the **dependent** variable.

This upgrades to arbitrary number of intermediate and independent variables in the natural way:

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

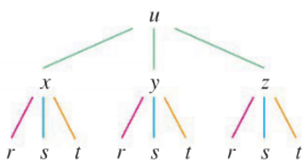


FIGURE 4

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s \sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

Chain Rule

Suppose $F(x, y) = 0$ and assume that y is (at least locally) a function of x.

Differentiating both sides w.r.t. x gives $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

and we obtain a slick formula for $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$ (Cheat sheet material)

At first, this seems a bit odd way to do things, but it deepens our understanding of some of our techniques for implicit differentiation, and shortens up a lot of the repetitive work involved in using implicit differentiation to say things about curves that are *not* functions.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

$$F(x, y, z) = 0$$

$$z = f(x, y)$$

$$F(x, y, f(x, y)) = 0$$

Here, we do not assume that y is locally a function of x.

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3$, $x = s\cos t$, $y = s\sin t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \iff z_s = z_x x_s + z_y y_s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \iff z_t = z_x x_t + z_y y_t$$

$$\boxed{\begin{aligned} z_s &= 2xy^3\cos t + 3x^2y^2\sin t \\ z_t &= (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) \end{aligned}}$$

27–30 Use Equation 6 to find dy/dx.

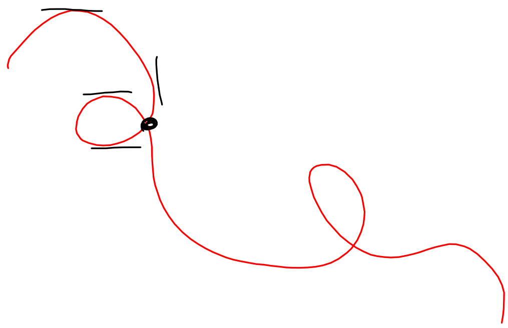
27. $\sqrt{xy} = 1 + x^2y$ #27 NA

28. $y^5 + x^2y^3 = 1 + ye^{x^2}$

#27 $(xy)^{\frac{1}{2}} = x^2y + 1$

$$\boxed{6} \quad \frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \implies F(x,y) = (xy)^{\frac{1}{2}} - x^2y - 1 = 0$$

$$\implies \frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{\frac{1}{2}(xy)^{-\frac{1}{2}}(y) - 2xy}{\frac{1}{2}(xy)^{-\frac{1}{2}}(x) - x^2} = \frac{dy}{dx}}$$



45–48 Assume that all the given functions are differentiable.

45. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Stewart has a way of referring back to these exercises for future exercises or to get some of the work involved in proving theorems out of the way, so they can just point you back to previous work.

Very underhanded.

45. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

48. If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t} \iff (z_x)^2 - (z_y)^2 = z_s z_t$$

$z = f(x(s,t), y(s,t))$

$z_s = z_x x_s + z_y y_s \implies z_x = \dfrac{z_s - z_y y_s}{x_s}$

$z_t = z_x x_t + z_y y_t \qquad z_y = \dfrac{z_s - z_x x_s}{y_s}$

$$\cancel{z_x^2 - z_y^2 = \frac{z_s^2 - 2 z_s z_y y_s \; {\color{red}+ z_y^2 y_s^2}}{x_s^2} - \frac{z_s^2 - 2 z_s z_x x_s + z_x^2 x_s^2}{y_s^2}}$$

$$\cancel{= \frac{z_s^2 y_s^2 - 2 z_s z_y y_s y_s^2 - z_s x_s^2 - 2 z_s z_x x_s x_s^2}{x_s^2 y_s^2}}$$

$z_s = z_x x_s + z_y y_s = z_x + z_y \qquad x = s+t,\ y = s-t$

$z_t = z_x x_t + z_y y_t = z_x - z_y$

$\implies z_s z_t = (z_x + z_y)(z_x - z_y) = z_x^2 - z_y^2$

use what's given, Steve!