Section 14.4 Tangent Planes and Linear Approximations

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\langle A, B, C \rangle = \vec{n}$$

$$\boxed{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

$$z = a(x - x_0) + b(y - y_0) + z_0$$

$$y - y_0 = m(x - x_0)$$

$$y = m(x - x_0) + y_0$$

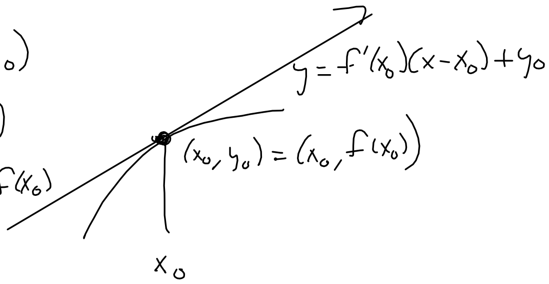
start @ (x_0, y_0) & use m to tell you how much y changes as you move away from x_0

Tangent Line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$= f'(x_0)(x - x_0) + f(x_0)$$

Linear Approx: $L(x) = f'(x_0)(x - x_0) + f(x_0)$



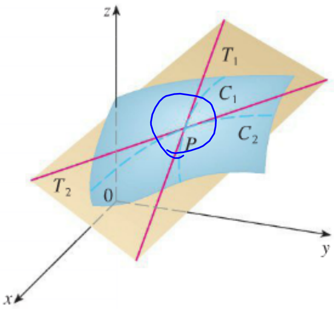


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Oblate spheroid

EARTH

$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

@ (x_0, y_0), $z = z_0$

TEC Visual 14.4 shows an animation of Figures 2 and 3.

ing the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

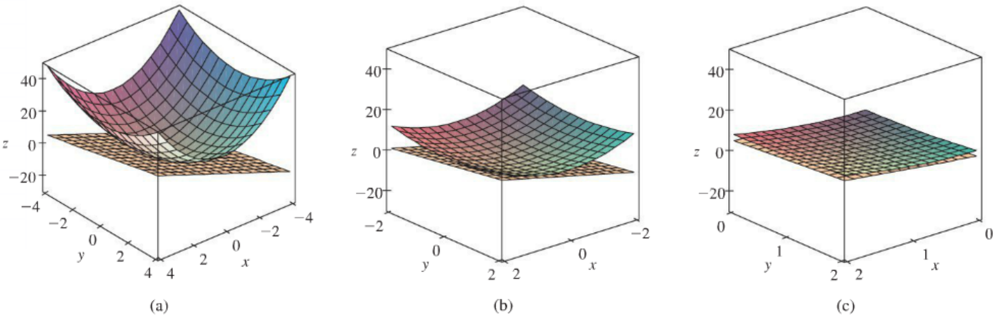
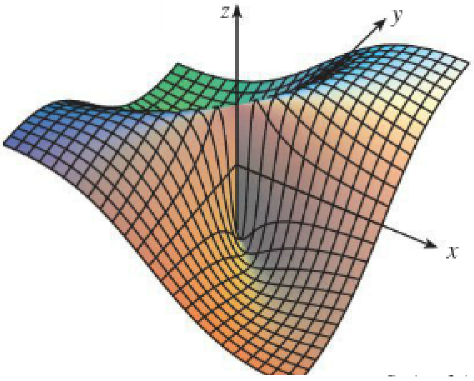


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Smooth curves and smooth surfaces are LOCALLY LINEAR.



This one has a cusp at the origin, its derivatives of all orders exist, but they aren't continuous at the origin.

So a function of two variables can behave badly even though both of its partial derivatives exist.

FIGURE 4

$f(x, y) = \dfrac{xy}{x^2 + y}$ if $(x, y) \neq (0, 0)$,

$f(0, 0) = 0$

Increment of y:

5 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$dy = f'(a)dx \approx \Delta y$

Increment of z:

6 $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

If you want to play with these ideas (and formalisms), then #46 is the bomb.

If you *don't*, then the following is a very practical way to check for differentiability is given by:

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

& continuity is 99% DOMAIN
(Maybe check some boundaries.)

Differentials in the Plane:

9 $$\Delta y \approx dy = f'(x)\,dx$$

In practice, $dx \equiv \Delta x$

The Differential of a surface in 3-space:

10 $$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

Also called the "total differential."

$\Delta z = f(x,y) - f(x_0,y_0)$

Actual change in height

Comes from tangent plane

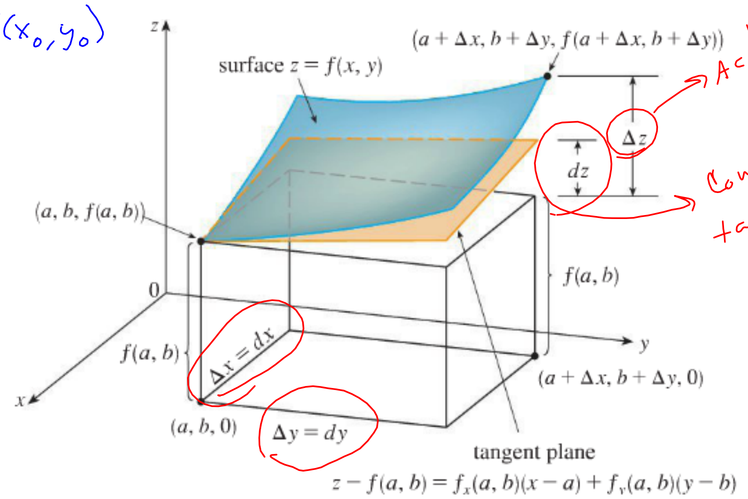


FIGURE 7

$g(x,y) = 6 - x - x^2 - 2y^2$

$g_x = -1 - 2x \Rightarrow g_x(1,2) = -1 - 2 = -3$

$g_y = -4y \Rightarrow g_y(1,2) = -8$

$g(1,2) = -4$

$L_{(1,2)}(x,y) = g_x(x-1) + g_y(y-2) - 4$

$= \boxed{-3(x-1) - 8(y-2) - 4 = L(x,y)}$

1–6 Find an equation of the tangent plane to the given surface at the specified point.

4. $z = x/y^2$, $(-4, 2, -1)$

11–16 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = 1 + x\ln(xy - 5)$, $(2, 3)$

12. $f(x, y) = \sqrt{xy}$, $(1, 4)$

13. $f(x, y) = x^2e^y$, $(1, 0)$

14. $f(x, y) = \dfrac{1 + y}{1 + x}$, $(1, 3)$

15. $f(x, y) = 4\arctan(xy)$, $(1, 1)$

16. $f(x, y) = y + \sin(x/y)$, $(0, 3)$

25–30 Find the differential of the function.

25. $z = e^{-2x} \cos 2\pi t$

26. $u = \sqrt{x^2 + 3y^2}$

31. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

Do this one, Steve.

32. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

33. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

34. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

39. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

$dR \approx \Delta R$

$$R^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1}$$

$$\frac{d}{dR_1}\left[R^{-1}\right] = -R^{-2}\frac{dR}{dR_1} = -R_1^{-2} \Longrightarrow$$

$$\frac{\partial R}{\partial R_1} = \frac{-R_1^{-2}}{-R^{-2}} = \frac{R^2}{R_1^2} = \frac{\left(R_1^{-1}+R_2^{-1}+R_3^{-1}\right)^2}{R_1^2}$$

$$\Longrightarrow dR = \frac{\partial R}{\partial R_1}dR_1 + \frac{\partial R}{\partial R_2}dR_2 + \frac{\partial R}{\partial R_3}dR_3$$

$$\left.\begin{array}{l} R_1 = 25\,\Omega,\ (.005)(25) = .125 = dR_1 \\ R_2 = 40\,\Omega,\ (.005)(40) = .2 = dR_2 \\ R_3 = 50\,\Omega,\ (.005)(50) = .25 = dR_3 \end{array}\right\} \Longrightarrow$$

$$dR = \left(R_1^{-1}+R_2^{-1}+R_3^{-1}\right)^{-2}\left[\left(R_1^{-2}\right)(.125) + \left(R_2^{-2}\right)(.2) + \left(R_3^{-2}\right)(.25)\right]$$

$$= \left(\frac{1}{25}+\frac{1}{40}+\frac{1}{50}\right)^{-2}\left[\left(\frac{1}{25}\right)^2(.125) + \left(\frac{1}{40}\right)^2(.2) + \left(\frac{1}{50}\right)^2(.25)\right]$$

$$\boxed{\approx .0588235294}$$

My problem was putting $\frac{1}{R}$ instead of R, right here

Need the -2 power!