Section 14.4 Tangent Planes and Linear Approximations

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

1 $$z - z_0 = a(x - x_0) + b(y - y_0)$$

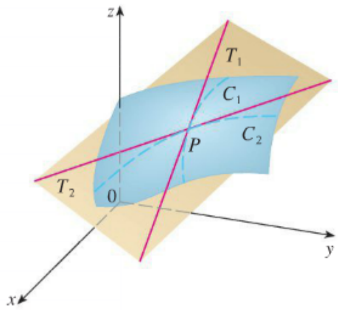


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Recall tangent lines in the plane:

TEC Visual 14.4 shows an animation of Figures 2 and 3.

ing the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

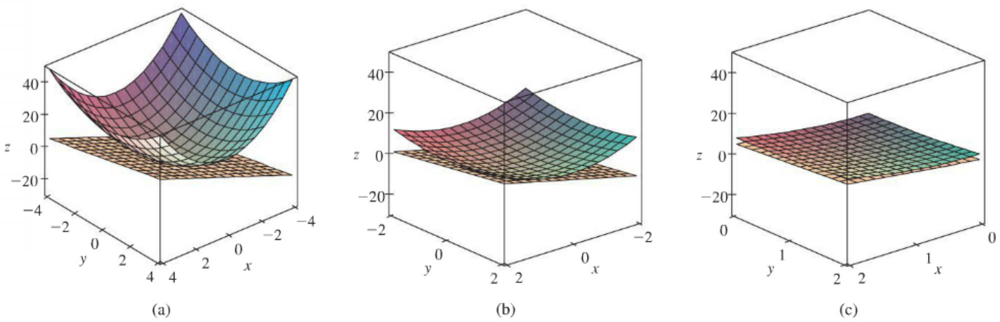
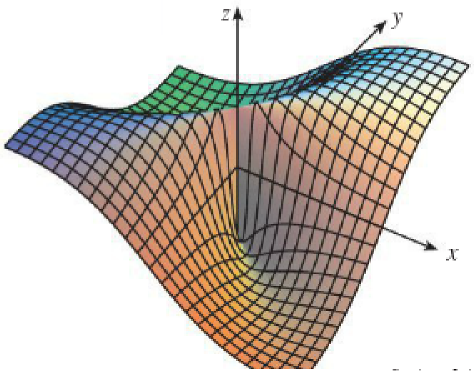


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).



This one has a cusp at the origin, its derivatives of all orders exist, but they aren't continuous at the origin.

So a function of two variables can behave badly even though both of its partial derivatives exist.

FIGURE 4
$f(x, y) = \dfrac{xy}{x^2 + y}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

Increment of y:

5 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Increment of z:

6 $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

If you want to play with these ideas (and formalisms), the #46 is the bomb.

If you *don't*, then the following is a very practical way to check for differentiability is given by:

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Differentials in the Plane:

9 $$dy = f'(x)\,dx$$

The Differential of a surface in 3-space:

10 $$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Also called the "total differential."

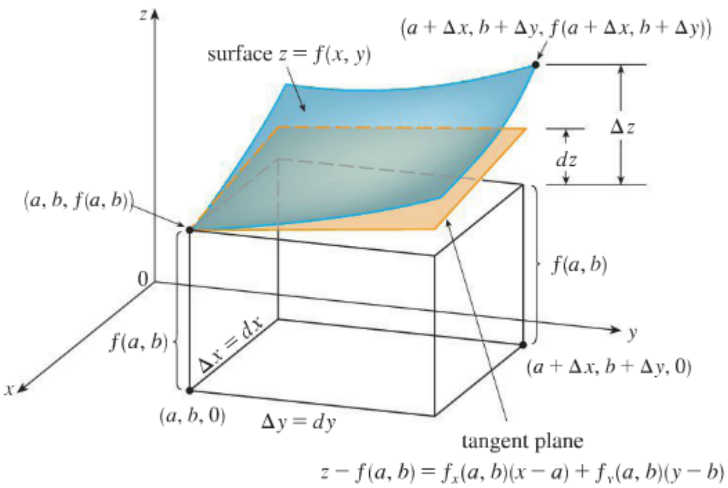


FIGURE 7

36. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

Treat this similar to #20, below, but I include some other stuff (like the plane $x = 1$, the trace of this plane on the surface, and the tangent to the surface living in this plane.

$g(x,y) = 6 - x - x^2 - 2y^2 \Rightarrow g_x = -1-2x \Rightarrow g_x(1,2) = -1-2 = -3 = g_x(1,2)$

$L_{(1,2)}(x,y)$

$g_y = -4y \Rightarrow g_y(1,2) = -4(2) = -8$

g

$g(1,2) = 6 - 1 - 1^2 - 2(2)^2 = 6-2-8 = -4 = g(1,2) = z_0$

$z = g_x(1,2)(x-1) + g_y(1,2)(y-2) - 4 =$

$L_{(1,2)}(x,y) = 3(x-1) - 8(y-2) - 4$

Trace of the plane $x=1$

$g(1,y) = 6-1-1-2y^2 = 4-2y^2 = \text{gtrace1}$

Space curve: $\text{gtrace1} = \langle 1, y, 4-2y^2 \rangle$

$= \langle 1, t, 4-2t^2 \rangle$ in Maple.

$g_y(1,2) = -8$

$x=1,\ y=y,\ z=-8(y-2)$

$x=1,\ y=t,\ z=-8(t-2) \rightsquigarrow \langle 1, t, -8(t-2) \rangle$

20. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing f and the tangent plane.

25–30 Find the differential of the function.

25. $z = x^3 \ln(y^2)$ #25 NA **26.** $v = y \cos xy$