Section 14.3 Partial Derivatives

§14.3 #s 4, 10, 11, 13, 15, 18, 21, 26, 29, 30, 47, 50, 52, 53, 56, 59, 71, 78, 81, 83

* Clairaut.

$f_x(a,b) = f_x(a, b) = g'(a)$ where $g(x) = f(x, b)$

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a,b) = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$D_x f$ $\quad \partial$ = "del" $\quad z = f(x, y)$

$$f_x = f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y = f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

$$f(x,y) = x^2 \sin(xy) + x^4 y^5$$ Find $f_x(x,y)$, $f_y(x,y)$

$f_x = 2x \sin(xy) + x^2 \cos(xy) \cdot y + 3x^3 y^5$

$f_y = x^2 \cos(xy) \cdot x + 5x^4 y^4$

$\frac{\partial}{\partial x}[xy] = y$

$\frac{d}{dx}[7x] = 7$

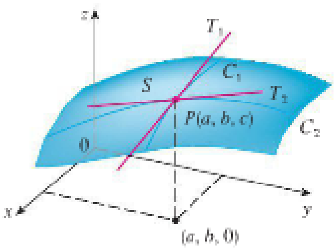


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

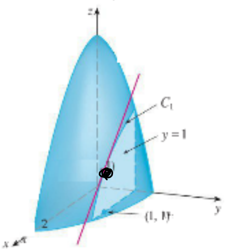
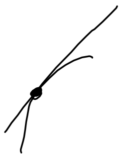


FIGURE 2



f_x vertical plane ∥ to xz-plane

f_y vertical plane ∥ to yz-plane.

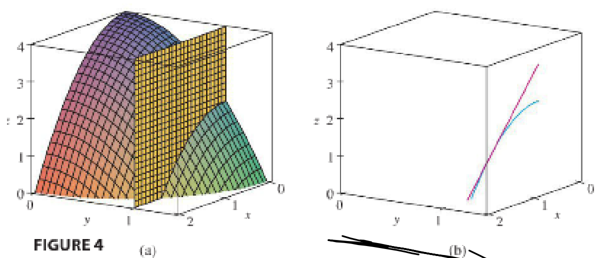


FIGURE 4

Functions of 3 or more variables...

$f(x_1, x_2, \ldots, x_n)$

f_{x_3}, etc. Same deal.

Higher Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

(order is reversed from f_{xy} notation

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f(x,y) = x^2 \sin(xy) + x^4 y^5$$

Find $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

$f_x = 2x\sin(xy) + x^2(\cos(xy))\cdot y + 4x^3y^5 = 2x\sin(xy) + \boxed{x^2 y\cos(xy)} + 4x^3y^5$

$f_{xx} = 2\sin(xy) + 2x(-\cos(xy))\cdot y + 12xy^5$ $= 2\sin(xy) + 2x(-\cos(xy))\,y + 2xy\cos(xy) + x^2y(-\sin(xy))\cdot y$

$f_{xy} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = 2x(\cos(xy))\cdot x + 20x^3y^4$

$f_y = x^2(\cos(xy))\cdot x + 5x^4y^4 = x^3\cos(xy) + 5x^4y^4$

$f_{yy} =$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Smoothness is why.

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **harmonic functions**;

heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$