I was sure I'd worked one like this, but it was a *similar* exercise that wasn't asking about velocity or speed.

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t.

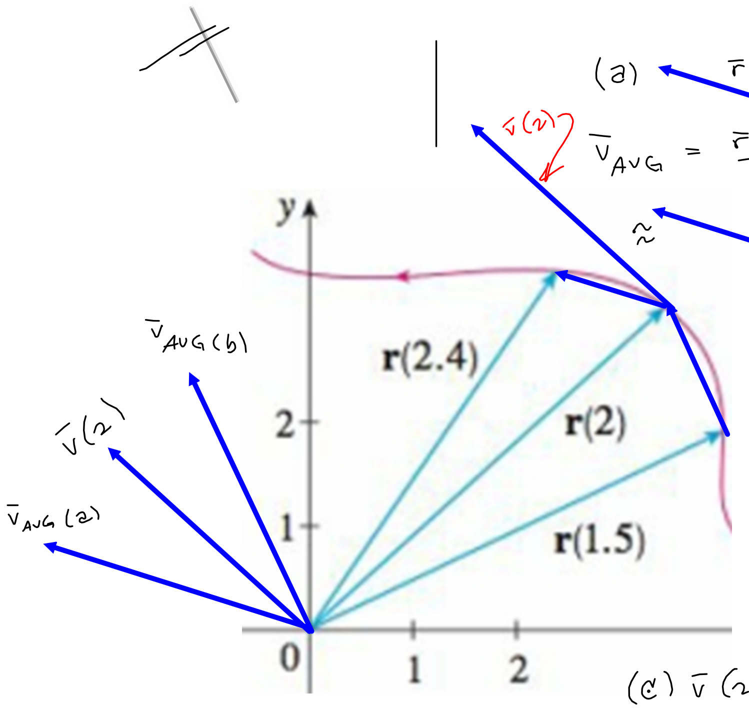
(a) Draw a vector that represents the average velocity of the particle over the time interval $2 \leqslant t \leqslant 2.4$.

(b) Draw a vector that represents the average velocity over the time interval $1.5 \leqslant t \leqslant 2$.

(c) Write an expression for the velocity vector $\mathbf{v}(2)$.

(d) Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.

1 unit



(a) $\bar{r}(2.4) - \bar{r}(2) \Rightarrow \bar{v}_{AVG} = \dfrac{\bar{r}(2.4) - \bar{r}(2)}{.4} = \dfrac{5}{2}\left(\bar{r}(2.4) - \bar{r}(2)\right)$

$\approx$ $\approx \bar{v}_{AVG}(a)$

(b) $\bar{r}(2) - \bar{r}(1.5) \approx$

$\Rightarrow \dfrac{\bar{r}(2) - \bar{r}(1.5)}{.5}$

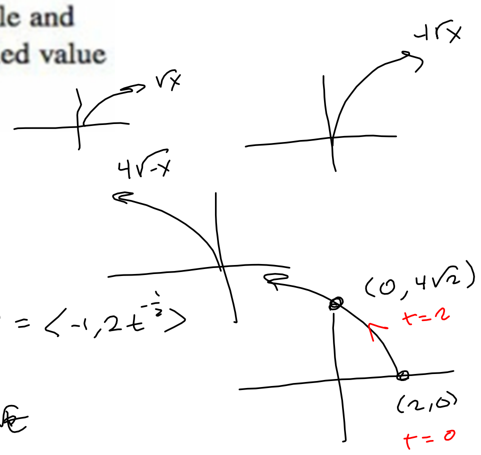
$\bar{v}_{AVG}(b)$

(c) $\bar{v}(2) = \lim\limits_{h \to 0} \dfrac{\bar{r}(2+h) - \bar{r}(2)}{h}$

(d) See $\bar{v}(2)$, above. I guesstimate the speed to be $\boxed{s(2) \approx 2.5}$

3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

4. $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle, \quad t = 1$



$x = 2 - t \implies t = 2 - x$

$\implies y = 4\sqrt{t} = 4\sqrt{2-x} = 4\sqrt{-(x-2)}$

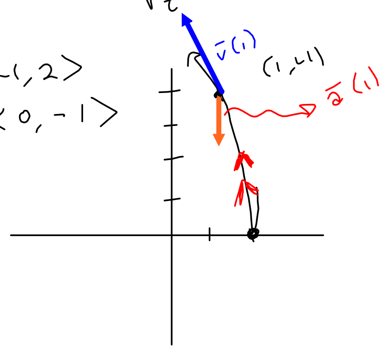
$\implies \bar{v}(t) = \text{velocity} = \bar{r}'(t) = \langle -1, \frac{2}{\sqrt{t}} \rangle = \langle -1, 2t^{-\frac{1}{2}} \rangle$

$\bar{v}(t) = \langle -1, 2t^{-\frac{1}{2}} \rangle \implies \quad t^{\frac{1}{2}} \xrightarrow{\frac{d}{dt}} \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

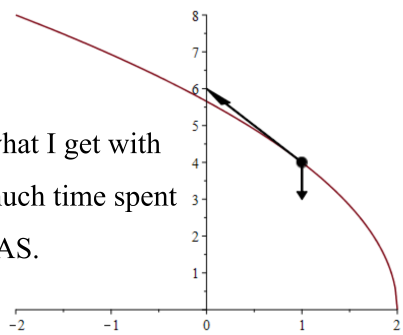
$\bar{a}(t) = \langle 0, -t^{-\frac{3}{2}} \rangle = \langle 0, \frac{-1}{\sqrt{t^3}} \rangle =$ acceleration.

$\implies \bar{v}(1) = \langle -1, 2 \rangle$

$\bar{a}(1) = \langle 0, -1 \rangle$



This is what I get with far too much time spent with a CAS.



Graph of the curve represented parametrically by the components of the given vector.

8. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

$\bar{r}(t) = \langle t, 2\cos(t), \sin(t) \rangle$ @ $t = 0$

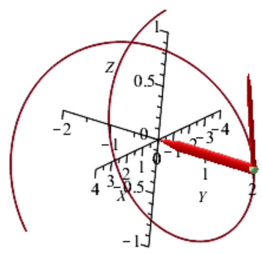
"Elliptical Helix"

$\bar{r}'(t) = \bar{v}(t) = \langle 1, -2\sin(t), \cos(t) \rangle$

$\implies \bar{r}''(t) = \bar{a}(t) = \bar{v}'(t) = \langle 0, -2\cos(t), -\sin(t) \rangle$

$\implies \bar{r}(0) = \langle 0, 2, 0 \rangle$

$$\left.\begin{aligned} \bar{r}'(0) &= \langle 1, 0, 1 \rangle \\ \bar{r}''(0) &= \langle 0, -2, 0 \rangle \end{aligned}\right\} \text{orthogonal!}$$



Graph of the curve represented parametrically by the components of the given vector.

9–14 Find the velocity, acceleration, and speed of a particle with the given position function. **11.** $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^{t}\,\mathbf{j} + e^{-t}\,\mathbf{k}$

Start w/ Lexicon :

(writing to communicate)

Let $\bar{v}(t)$ = velocity, $\bar{a}(t)$ = acceleration, and $s(t)$ = SPEED

$\bar{r}(t)$ = position $= \langle \sqrt{2}t, e^{t}, e^{-t} \rangle$

$\Rightarrow \bar{r}'(t) = \boxed{\bar{v}(t) = \langle \sqrt{2}, e^{t}, -e^{-t} \rangle}$

$\Rightarrow \bar{r}''(t) = \bar{v}'(t) = \boxed{\bar{a}(t) = \langle 0, e^{t}, e^{-t} \rangle}$

$\Rightarrow \boxed{s(t) = \| \bar{v}(t) \| = \sqrt{\sqrt{2}^{2} + e^{2t} + e^{-2t}}}$

15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15. $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j}$, $\mathbf{v}(0) = \mathbf{k}$, $\mathbf{r}(0) = \mathbf{i}$ Standard Exercise in Differential Equations.

$\bar{v}(t) = \int \bar{a}(t)\,dt + \bar{C}$ (That constant will turn out to be $\bar{v}(0)$)

$= \int \langle 1, 2, 0 \rangle\,dt + \bar{C}$

$= \langle t, 2t, 0 \rangle + C \Rightarrow \bar{v}(0) = \langle 0,0,0 \rangle + \bar{C} = \langle 0,0,1 \rangle$

$\Rightarrow \bar{C} = \langle 0,0,1 \rangle$

$\Rightarrow \bar{v}(t) = \langle t, 2t, 0 \rangle + \langle 0,0,1 \rangle = \boxed{\langle t, 2t, 1 \rangle = \bar{v}(t)}$

$\Rightarrow \bar{r}(t) = \int \bar{v}(t)\,dt + \bar{D} = \int \langle t, 2t, 1 \rangle\,dt + \bar{D}$

$= \langle \frac{1}{2}t^{2}, t^{2}, t \rangle + \bar{D} \Rightarrow$

$\bar{r}(0) = \langle 0,0,0 \rangle + \bar{D} = \langle 1,0,0 \rangle = \bar{D}$

$\Rightarrow \boxed{\bar{r}(t) = \langle \frac{1}{2}t^{2} + 1, t^{2}, t \rangle}$

19. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

$\Longrightarrow \vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle = \vec{v}(t) \Longrightarrow$

$\|\vec{r}\,'(t)\| = \text{speed} = \|\vec{v}(t)\|$

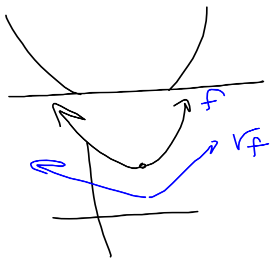
$= \left((2t)^2 + 5^2 + (2t-16)^2 \right)^{\frac{1}{2}} = s(t) = \text{speed function}$

$= \left(4t^2 + 25 + 4t^2 - 64t + 256 \right)^{\frac{1}{2}}$

$= \left(8t^2 - 64t + 271 \right)^{\frac{1}{2}}$ Minimize!

$= \left(8(t^2 - 8t \quad) + 271 \right)^{\frac{1}{2}}$

shows its vertex $= \left(\frac{8}{2}, ?\right) = (4, ?)$



Calc I tools:

$s'(t) = \frac{1}{2}\left(8t^2 - 64t + 271\right)^{-\frac{1}{2}}(16t - 64)$

$= \dfrac{16(t-4)}{2\sqrt{8t^2-64t+271}}$ SET 0 $\Longrightarrow t = 0$

Critical points from Denom $= 0$

i.e., $8t^2 - 64t + 271 = 0$

$8(t^2 - 8t \quad) = -271$

Never zero.

$b^2 - 4ac =$

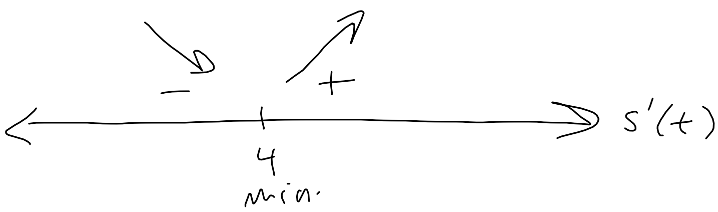
$64^2 - 4(8)(271) < 0$

$32(2 \cdot 64 - 271) < 0$

Nevah!

$\Longrightarrow t = 4$ is only crit. value

$\Longrightarrow$ $\boxed{s(4) = \text{min.}}$



23. A projectile is fired with an initial speed of 500 m/s and angle of elevation 30°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

33–38 Find the tangential and normal components of the acceleration vector. **33.** $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 3t^2\,\mathbf{j}$

36. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}$

42. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$F = m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e \qquad \frac{d^2\bar{r}}{dt^2}$$

$F = ma$ Newton's 2nd

(a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\frac{m(0)}{m(t)}\mathbf{v}_e$.

$$kg \cdot \frac{m}{s^2} = \frac{kg}{s} \cdot \frac{m}{s} = \text{Newtons.}$$

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

Rocket Science!

Mass per unit time times velocity is force! That's how rocket engines work!

$m_0 \equiv m(0)$, $\bar{v}_0 = \bar{v}(0)$

(a) $m\frac{dv}{dt} = m(t)\frac{d\bar{v}(t)}{dt} = \frac{d}{dt}[m(t)]\,\bar{v}_e$

$$\int \frac{d\bar{v}(t)}{dt}\,dt = \int \frac{\frac{dm(t)}{dt}}{m(t)}\bar{v}_e\,dt + \bar{c}$$

$$\int d\bar{v}(t) = \bar{v}(t) = \int \bar{v}_e\frac{dm}{m} + \bar{c}$$

$$\boxed{\bar{v}(t) = \bar{v}_e \ln(m(t)) + \bar{c}}$$

$\Rightarrow \bar{c} = \bar{v}(t) - \bar{v}_e \ln(m(t))$

$\Rightarrow \boxed{\bar{c} = \bar{v}(0) - v_e \ln(m_0)}$

$\bar{v}(t) = \bar{v}_e \ln(m(t)) + \bar{v}_0 - v_e \ln(m_0)$

$= \bar{v}_0 + \bar{v}_e \ln\left(\frac{m(t)}{m_0}\right) = \text{Fine} = \boxed{\bar{v}(0) - \bar{v}_e \ln\left(\frac{m_0}{m(t)}\right) = \bar{v}(t)}$

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

$\bar{v}(0) = \bar{0}$!

$\bar{v} = v$ is OK.

$v(t) = \bar{v}(0) - \bar{v}_e \ln\left(\frac{m_0}{m(t)}\right)$

$= -\bar{v}_e \ln\left(\frac{m_0}{m(t)}\right) \overset{\text{SET}}{=} -2\bar{v}_e$

$\Rightarrow \ln\left(\frac{m_0}{m(t)}\right) = 2$

$e^{\sim} = e^2$

$\frac{m_0}{m(t)} = e^2 \Rightarrow$

$m(t) = \frac{m_0}{e^2} = \frac{1}{e^2}m_0$ is mass of rocket + fuel, and so the amount of fuel burned would be

$m_0 - \frac{m_0}{e^2} = \frac{e^2 m_0 - m_0}{e^2} = \frac{e^2-1}{e^2}m_0$

$\Rightarrow \frac{e^2-1}{e^2}$ of the mass of the rocket + fuel you started with was burned as fuel.