Recall: Arc Length for $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$

from $t = \alpha$ to $t = \beta$

$$L = s = \int_\alpha^\beta \underbrace{\sqrt{x'^2 + y'^2 + z'^2}\, dt}_{ds}$$

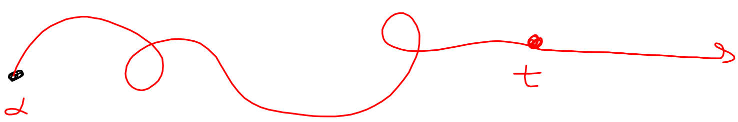
$$s = s(t) = \int_\alpha^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\, du = \int_\alpha^t \|\bar{r}'(u)\|\, du$$

$$\Longrightarrow \frac{ds}{dt} = s'(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \boxed{= \|\bar{r}'(t)\| = \frac{ds}{dt}}$$

$\boxed{D}$ $\kappa = \left\| \frac{d\bar{T}}{ds} \right\|$. If $t = t(s)$ is function of s, then chain rule:

$$\frac{d\bar{T}}{dt} = \frac{d\bar{T}}{ds} \cdot \frac{ds}{dt}, \text{ so}$$

$$\left\| \frac{d\bar{T}}{ds} \right\| = \left\| \frac{\frac{d\bar{T}}{dt}}{\frac{ds}{dt}} \right\| = \frac{\|\bar{T}'(t)\|}{\|\bar{r}'\|}$$



$\boxed{T10}$ $\kappa(t) = \dfrac{\|\bar r' \times \bar r''\|}{\|\bar r'\|^3} = \|\bar r'(t)\|$, etc. .

Proof $\bar T = \dfrac{\bar r'}{\|\bar r'\|} \Longrightarrow \bar r' = \|\bar r'\|\,\bar T =$

$$\Longrightarrow \bar r'' = \left(\|\bar r'\|\right)'\bar T + \|\bar r'\|\,\bar T'$$

$$= \left(\frac{ds}{dt}\right)'\bar T + \frac{ds}{dt}\bar T'$$

$$= \frac{d^2s}{dt^2}\bar T + \frac{ds}{dt}\bar T'$$

Consider $\bar r' \times \bar r''$

$$= \bar r' \times \left(\frac{d^2s}{dt^2}\bar T\right) + \bar r' \times \frac{ds}{dt}\bar T'$$

$\bar T = \dfrac{\bar r'}{\|\bar r'\|} \Longrightarrow$

$\bar r' = \|\bar r'\|\,\bar T$

$$= \left(\frac{d^2s}{dt^2}\right)\left(\bar r' \times \bar T\right) + \left(\frac{ds}{dt}\right)\bar r' \times \bar T'$$

$$= \underbrace{\boxed{\frac{d^2s}{dt^2}\left(\|\bar r'\|\,\bar T \times \bar T\right)}}_{=0!} + \left(\frac{ds}{dt}\right)\left(\bar r' \times \bar T'\right)$$

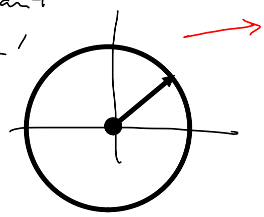
$\|\bar T \times \bar T\|$

$= \|\bar T\|^2 \sin\theta = 0$

$$= \left(\frac{ds}{dt}\right)\left(\bar r' \times \bar T'\right)$$

$\|\bar r\| =$ constant

$\Longrightarrow \bar r \perp \bar r'$



$$= \boxed{\|\bar r'\|\left(\|\bar r'\|\right.}\left.\bar T \times \bar T'\right)$$

$$= \|\bar r'\|^2\,\|\bar T\|\,\|\bar T'\|\sin\theta \quad \& \quad \bar T \perp \bar T'$$

$$= \left(\frac{ds}{dt}\right)^2 \|\bar T \times \bar T'\| = \|\bar r' \times \bar r''\|$$

$$\Longrightarrow \left(\frac{ds}{dt}\right)^2 \|\bar T'\| = \|\bar r' \times \bar r''\|$$

$$\Longrightarrow \|\bar T'\| = \frac{\|\bar r' \times \bar r''\|}{\left(\frac{ds}{dt}\right)^2} = \frac{\|\bar r' \times \bar r''\|}{\|\bar r'\|^2}$$

$$\Longrightarrow \kappa = \left|\frac{d\bar T}{ds}\right| = \frac{\frac{d\bar T}{dt}}{\frac{ds}{dt}} = \frac{\|\bar T\|}{\|\bar r'\|} = \frac{\frac{\|\bar r' \times \bar r''\|}{\|\bar r'\|^2}}{\|\bar r'\|}$$

$$\boxed{\kappa = \frac{\|\bar r' \times \bar r''\|}{\|\bar r'\|^3}}$$

An example similar to example 4.

Given $\bar{r}(t) = \langle \sin(\pi t), \cos(\pi t), t \rangle$. Re-parametrize $\bar{r}(t)$ by finding $t = t(s)$ as a function of s, starting at the point $\bar{r}(0) = \langle 0,1,0 \rangle$.

$$s(t) = \int_0^t \|\bar{r}'(u)\| du = \int_0^t \sqrt{x'^2 + y'^2 + z'^2} = \int_0^t \sqrt{(\pi \cos(\pi u))^2 + (-\pi \sin(\pi u))^2 + 1}\, du$$

$$= \int_0^t \sqrt{\pi^2 \cos^2(\pi u) + \pi^2 \sin^2(\pi u) + 1}\, du$$

$$= \int_0^t \sqrt{\pi^2 (\cos^2(\pi u) + \sin^2(\pi u)) + 1}\, du$$

$$= \int_0^t \sqrt{\pi^2+1}\, du = \sqrt{\pi^2+1} \int_0^t du = \sqrt{\pi^2+1}\, u \Big|_0^t = \sqrt{\pi^2+1}\, t = s$$

$$\Longrightarrow t = \frac{s}{\sqrt{\pi^2+1}} = t(s) = t \text{ as function of } s\,!$$

$$\bar{r}(t(s)) = \left\langle \sin\left(\pi \left(\frac{s}{\sqrt{\pi^2+1}}\right)\right), \cos\left(\pi \left(\frac{s}{\sqrt{\pi^2+1}}\right)\right), \frac{s}{\sqrt{\pi^2+1}} \right\rangle$$

1–6 Find the length of the curve.

1. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t \rangle, \quad -5 \leqslant t \leqslant 5$

$$S = \int_{-5}^{5} \|\bar{r}'\|\, dt = \int_{-5}^{5} \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\, dt$$

$$= \int_{-5}^{5} \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \int_{-5}^{5} \sqrt{10}\, dt$$

$$= \sqrt{10}\, t \Big|_{-5}^{5} = 5\sqrt{10} - (-5\sqrt{10}) = \boxed{10\sqrt{10} = S}$$

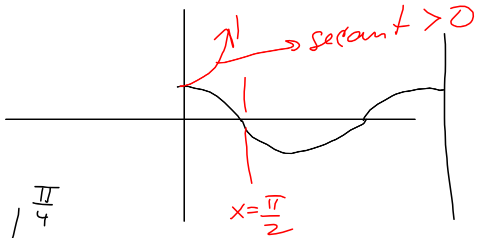
4. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln \cos t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/4$

$$\bar{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$$

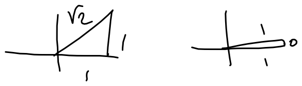
$$\Rightarrow \bar{r}'(t) = \left\langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \right\rangle$$

$$S = \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\, dt = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 t}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 t}\, dt = \int_0^{\frac{\pi}{4}} |\sec t|\, dt$$



$$= \int_0^{\frac{\pi}{4}} \sec t\, dt = \ln|\sec t + \tan t| \Big|_0^{\frac{\pi}{4}} = \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$



$$= \ln|\sqrt{2} + 1| - \ln|1|$$

$$\boxed{= \ln|\sqrt{2} + 1|}$$

$$\left(\frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} = \int \frac{du}{u}, \text{ where } u = \sec t + \tan t \right.$$

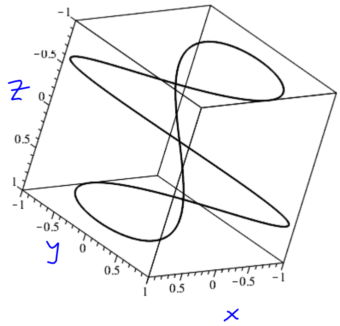
7–9 Find the length of the curve correct to four decimal places. (Use a calculator to approximate the integral.)

7. $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle, \quad 0 \leqslant t \leqslant 2$

$$\bar{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle \Rightarrow$$

$$S = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\, dt \approx 18.68329847 \approx \boxed{18.6833 \approx S}$$

10. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve correct to four decimal places.



$$\int_0^{2\pi} \|\bar{r}\,'(t)\|\,dt =$$

$$= \int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2(4t) + 9\cos^2(3t)}\,dt \approx 16.02640249$$

13–14 (a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P, and (b) find the point 4 units along the curve (in the direction of increasing t) from P.

13. $\mathbf{r}(t) = (5 - t)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + 3t\,\mathbf{k}$, $P(4, 1, 3)$

a) $\bar{r}(t) = \langle -t+5, 4t-3, 3t \rangle \Longrightarrow \bar{r}\,'(t) = \langle -1, 4, 3 \rangle$

$\bar{r}(1) = \langle 4, 1, 3 \rangle$ ✓

$$s(t) = \int_1^t \|\bar{r}\,'\|\,du = \int_1^t \sqrt{1^2+4^2+3^2}\,du = \int_1^t \sqrt{1+16+9}\,du = \sqrt{26}\int_1^t du$$

$$= \sqrt{26}\,u \Big|_1^t = \sqrt{26}\,t - \sqrt{26} = s \Longrightarrow \sqrt{26}\,t = s - \sqrt{26}$$

$$\Longrightarrow t = \frac{s-\sqrt{26}}{\sqrt{26}} = \frac{s}{\sqrt{26}} - 1$$

$$\Longrightarrow \bar{r}(t) = \left\langle -\left(\frac{s}{\sqrt{26}} - 1\right) + 5,\ 4\left(\frac{s}{\sqrt{26}} - 1\right) - 3,\ 3\left(\frac{s}{\sqrt{26}} - 1\right) \right\rangle$$

$$= \boxed{\left\langle \frac{-s}{\sqrt{26}} + 6,\ \frac{4s}{\sqrt{26}} - 7,\ \frac{3s}{\sqrt{26}} - 3 \right\rangle = r(s)} = r(t(s))$$

b) $$\boxed{\text{Find } \bar{r}\Big|_{s=4} = \left\langle -\frac{4}{\sqrt{26}} + 6,\ \frac{16}{\sqrt{26}} - 7,\ \frac{12}{\sqrt{26}} - 3 \right\rangle}$$

17–20

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

17. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \implies \bar{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle$

$$\implies \|\bar{r}'(t)\| = \sqrt{1^2 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

$$\implies \bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$$

$$\bar{T}'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle \qquad \|a\bar{v}\| = a\|\bar{v}\|$$

$$\implies \|\bar{T}'(t)\| = \frac{1}{\sqrt{10}}\sqrt{9\cos^2 t + 9\sin^2 t} = \frac{3}{\sqrt{10}}$$

$$\frac{\bar{T}'(t)}{\|\bar{T}'(t)\|} = \frac{\frac{1}{\sqrt{10}}\langle 0, -3\cos(t), -3\sin(t)\rangle}{\frac{3}{\sqrt{10}}} = \boxed{\frac{1}{3}\langle 0, -3\cos t, -3\sin t\rangle = \bar{N}(t)}$$

Always points to the inside of the curve.

Check out Learning Tool!

LINK !

b $\boxed{9}$ $$\kappa(t) = \frac{\|\bar{T}'(t)\|}{\|\bar{r}'(t)\|} = \frac{\frac{3}{\sqrt{10}}}{\|\langle 1, -3\sin t, 3\cos t\rangle\|} = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \boxed{3 = \kappa}$$

21–23 Use Theorem 10 to find the curvature.

21. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k}$

$$\kappa(t) = \frac{\|\bar r' \times \bar r''\|}{\|\bar r'\|^3}$$

$= \langle 0, t^3, t^2 \rangle$

$\Rightarrow \bar r' = \langle 0, 3t^2, 2t \rangle \Rightarrow \|\bar r'\| = \boxed{\sqrt{9t^4 + 4t^2}} = |t|\sqrt{9t^2+4} = t\sqrt{9t^2+4}$ If $t \ge 0$

$\bar r'' = \langle 0, 6t, 2 \rangle$

$$\Rightarrow \bar r' \times \bar r'' = \begin{array}{l} \langle 0, 3t^2, 2t \rangle, 0, 3t^2 \\ \langle 0, 6t, 2 \rangle, 0, 6t \\ \hline \langle -6t^2, 0, 0 \rangle \end{array} \Rightarrow \kappa(t) = \frac{\sqrt{36t^4}}{\sqrt{9t^4+4t^2}} = \frac{6t^2}{t\sqrt{9t^2+4}}$$

$$\boxed{\kappa(t) = \frac{6t}{\sqrt{9t^2+4}}} \quad (t \ge 0 \text{ assumed})$$

27–29 Use Formula 11 to find the curvature.

28. $y = \tan x$ $\boxed{11}$ $\bar r = \langle x, f(x) \rangle$ when $y = f(x)$ in 2D, so that

$$\kappa(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}$$

47–48 Find the vectors **T**, **N**, and **B** at the given point.

47. $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$, $(1, \frac{2}{3}, 1)$ $= \bar{r}(1)$

$$\bar{T}\Big|_{t=1} = \frac{\bar{r}'}{\|\bar{r}'\|}\Big|_{t=1} = \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2+4t^4+1}}\Big|_{t=1} = \frac{\langle 2,2,1\rangle}{3} = \boxed{\frac{1}{3}\langle 2,2,1\rangle = \bar{T}(1)}$$

$\bar{T}$

$$\bar{N}\Big|_{t=1} = \frac{\bar{T}'}{\|\bar{T}'\|}\Big|_{t=1} = \frac{\bar{r}''}{\|\bar{r}''\|}\Big|_{t=1} = \frac{\langle 2, 4t, 0\rangle}{\sqrt{4+16t^2}}\Big|_{t=1} = \frac{1}{2\sqrt{1+4t^2}}\langle 2, 4t, 0\rangle\Big|_{t=1} = \bar{N}(1) = \frac{1}{2}\langle 2,4,0\rangle$$

$$\boxed{= \langle 1, 2, 0\rangle = \bar{N}(1)}$$

$$\bar{B}(1) = \bar{T}(1) \times \bar{N}(1) = \begin{array}{r} \frac{1}{3}\langle 2,2,1\rangle, 2, 2 \\ \times \quad \langle 1,2,0\rangle, 1, 2 \\ \hline \end{array}$$

$$\boxed{\frac{1}{3}\langle -2, 1, 2\rangle = \bar{B}(1)}$$

49–50 Find equations of the normal plane and osculating plane of the curve at the given point.

Normal to the curve, i.e. $\bar{T}$ or $\bar{r}$ are orthog. to NORMAL to the Normal.

50. $x = \ln t$, $y = 2t$, $z = t^2$; $(0, 2, 1)$ $= \bar{r}(1)$

$\bar{r} = \langle \ln(t), 2t, t^2\rangle$

$\Rightarrow \bar{r}'(t) = \langle \frac{1}{t}, 2, 2t\rangle \Rightarrow \bar{r}'(1) = \langle 1, 2, 2\rangle = \bar{n}$

$\boxed{1(x-0)+2(y-2)+2(z-1)=0}$ Normal Plane.

$\bar{T}$ & $\bar{N}$ lie in the osculating plane

The $\bar{n}$ for this is $\bar{T}\times\bar{N}\Big|_{t=1}$

$$\bar{N} = \frac{\bar{T}'}{\|\bar{T}'\|} \quad : \quad \bar{T} = \frac{\bar{r}'}{\|\bar{r}'\|} = \frac{\langle \frac{1}{t}, 2, 2t\rangle}{\sqrt{\frac{1}{t^2}+2^2+4t^2}} \Rightarrow$$

$$\bar{T}(1) = \frac{\langle 1,2,2\rangle}{\sqrt{1+4+4}} = \frac{1}{3}\langle 1,2,2\rangle = \bar{T}(1)$$

Need $\bar{T}'$, man!

$$\frac{1}{\sqrt{\frac{1}{t^2}+4+4t^2}}\langle \tfrac{1}{t}, 2, 2t\rangle = \left(\tfrac{1}{t^2}+4+4t^2\right)^{-\frac{1}{2}}\langle \tfrac{1}{t}, 2, 2t\rangle$$

$$\frac{d}{dt}\left[f(t)\bar{v}(t)\right] = f'(t)\bar{v}(t) + f(t)\bar{v}'(t)$$

$$\Rightarrow \bar{T}'(t) = -\frac{1}{2}\left(\tfrac{1}{t^2}+4+4t^2\right)^{-\frac{3}{2}}\left(-2t^{-3}+8t\right)\langle \tfrac{1}{t}, 2, 2t\rangle$$

$$+ \left(\tfrac{1}{t^2}+4+4t^2\right)^{-\frac{1}{2}}\langle -\tfrac{1}{t^2}, 0, 2\rangle$$

$$\Rightarrow \bar{T}'(1) = -\frac{1}{2}(1+4+4)^{-\frac{3}{2}}\left(-\tfrac{2}{1^3}+8(1)\right)\langle 1,2,2\rangle + (1+4+4)^{-\frac{1}{2}}\langle -1, 0, 2\rangle$$

$$= -\frac{1}{2}\left(\left(9\right)^{\frac{1}{2}}\right)^{-3}(6)\langle 1,2,2\rangle + \frac{1}{3}\langle -1,0,2\rangle$$

$$= -\frac{1}{54}(6)\langle 1,2,2\rangle + \langle -\tfrac{1}{3}, 0, \tfrac{2}{3}\rangle$$

$$= -\frac{1}{9}\langle 1,2,2\rangle + \langle -\tfrac{1}{3}, 0, \tfrac{2}{3}\rangle$$

$$= \langle -\tfrac{1}{9}, -\tfrac{2}{9}, -\tfrac{2}{9}\rangle + \langle -\tfrac{3}{9}, 0, \tfrac{6}{9}\rangle$$

$$= \langle -\tfrac{4}{9}, -\tfrac{2}{9}, \tfrac{4}{9}\rangle = -\frac{1}{9}\langle 4, 2, 4\rangle$$

$$\Rightarrow \bar{N}(1) = \frac{-\frac{1}{9}\langle 4,2,4\rangle}{\sqrt{\frac{16+4+16}{81}}} = \frac{-\frac{1}{9}\langle 4,2,4\rangle}{\frac{6}{9}} = \frac{9}{6}\cdot\frac{1}{9}\langle 4,2,4\rangle$$

$$\boxed{= \frac{1}{6}\langle 4,2,4\rangle = \bar{N}(1)}$$

$$= \frac{1}{3}\langle 2,1,2\rangle$$

$$\bar{T}\times\bar{N} = \frac{1}{3}\langle 1,2,2\rangle \times \frac{1}{3}\langle 2,1,2\rangle :$$

$$\begin{array}{r} \frac{1}{3}\langle 1,2,2\rangle, \cancel{1, 2} \\ \times\ \frac{1}{3}\langle 2,1,2\rangle, \cancel{2, 1} \\ \hline \frac{1}{9}\langle 2, 2, -3\rangle = \bar{B} \end{array}$$

is normal to the osculating plane

$(0,2,1)$

$$= \langle \tfrac{2}{9}, \tfrac{2}{9}, -\tfrac{1}{3}\rangle$$

$$9\bar{B} = \langle 2, 2, -3\rangle$$

$\boxed{2(x-0)+2(y-2)-3(z-1)=0 \text{ is eq'n of osculating plane}}$