

13.1 Vector Functions and Space Curves

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left(\lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right)$$

provided the limits of the component functions exist.

A vector function \mathbf{r} is **continuous at a** if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$$

Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f , g , and h are continuous real-valued functions on an interval I . Then the set C of all points (x, y, z) in space, where

$$\boxed{2} \quad x = f(t) \quad y = g(t) \quad z = h(t)$$

and t varies throughout the interval I , is called a **space curve**. The equations in (2) are called **parametric equations of C** and t is called a **parameter**.

EXAMPLE 3 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

$$x = 1 + t, \quad y = 2 + 5t, \quad z = -1 + 6t$$

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}_0 + t \bar{\mathbf{v}}$$

Line in 3-space.

$$= \langle 1, 2, -1 \rangle + t \langle 1, 5, 6 \rangle \text{ is its vector equation.}$$

EXAMPLE 4 Sketch the curve whose vector equation is

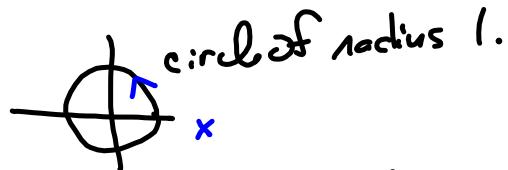
$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

projection into xy-plane:

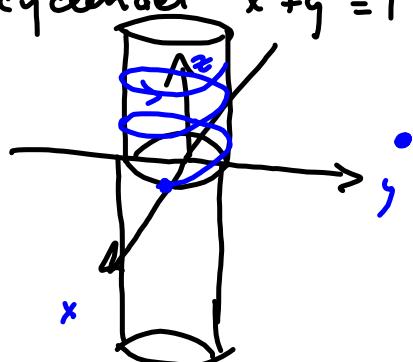
Points on the unit circle &

$x = \cos(t)$, $y = \sin(t)$ are parametric equations for the unit circle

$$\mathbf{r} = \langle \cos t, \sin t \rangle$$



So as z varies, it makes a spiral along the cylinder $x^2 + y^2 = 1$



EXAMPLE 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



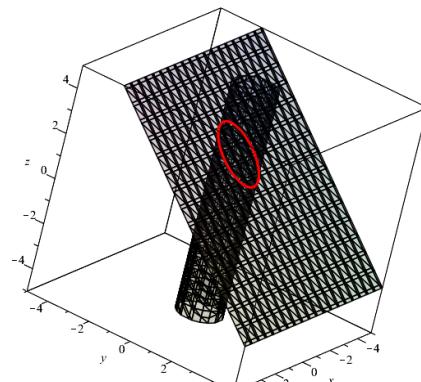
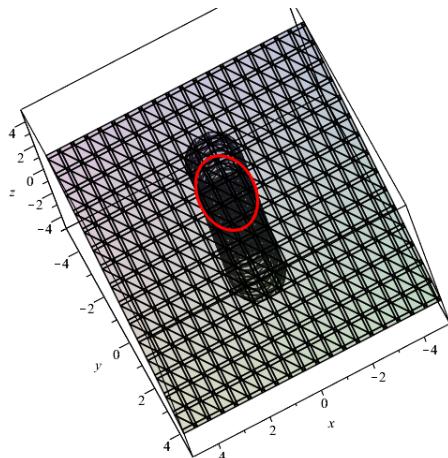
$$\langle x(t), y(t) \rangle = \langle \cos(t), \sin(t) \rangle \text{ in 2-D (3-D it's a cylinder)}$$

$$y + z = 2 \implies$$

$$\sin(t) + z = 2$$

$$z = 2 - \sin(t) \implies$$

$$\vec{r} = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle$$



1-2 Find the domain of the vector function.

$$1. \mathbf{r}(t) = \left\langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$$

$$D = (-1, 3)$$

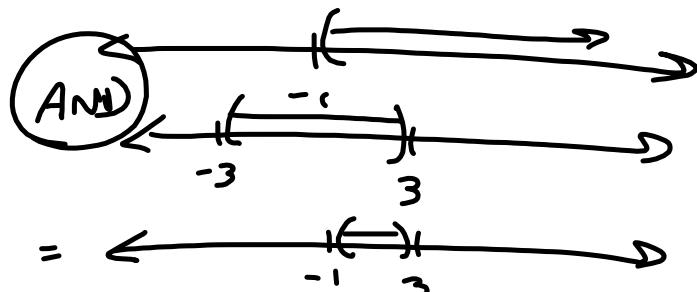
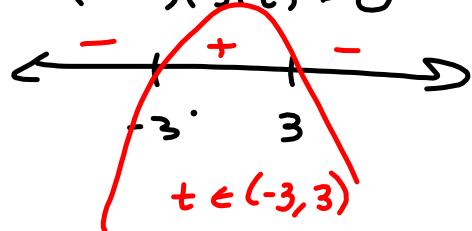
$$\ln(t+1) \quad \text{Need } t+1 > 0$$

$$\dots \quad t > -1$$

$$\frac{t}{\sqrt{9-t^2}}$$

$$\text{Need } 9-t^2 > 0$$

$$(3-t)(3+t) > 0$$

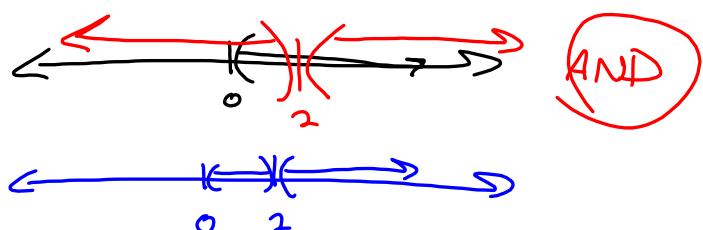


$$2. \mathbf{r}(t) = \cos t \mathbf{i} + \ln t \mathbf{j} + \frac{1}{t-2} \mathbf{k}$$

$$= \left\langle \cos(t), \ln(t), \frac{1}{t-2} \right\rangle$$

$$\mathbb{R} \quad t > 0 \quad t \neq 2$$

$$D = (0, 2) \cup (2, \infty)$$



3-6 Find the limit.

3. $\lim_{t \rightarrow 0} \left(e^{-3t} \mathbf{i} + \frac{t^2}{\sin^2 t} \mathbf{j} + \cos 2t \mathbf{k} \right)$

$$\vec{r}(t) = \langle e^{-3t}, \frac{t^2}{\sin^2 t}, \cos(2t) \rangle$$

$$e^{-3t} \xrightarrow[t \rightarrow 0]{} e^{-3(0)}^* = e^0 = 1$$

$$\frac{t^2}{\sin^2 t} \xrightarrow[t \rightarrow 0]{} \left(\frac{t}{\sin(t)} \right)^2 \xrightarrow[t \rightarrow 0]{} 1^2 = 1$$

FACT $\frac{\sin t}{t} \xrightarrow[t \rightarrow 0]{} 1$ (In proof of $\frac{d}{dx}(\sin(x)) = \cos(x)$)

$$\cos(2t) \xrightarrow[t \rightarrow 0]{} \cos(2(0))^* = \cos(0) = 1$$

*These are cuts so $\lim_{t \rightarrow 0} f(t) = f(0)$, by def'n.

$$\boxed{\langle 1, 1, 1 \rangle = \lim_{t \rightarrow 0} \vec{r}(t)}$$

5. $\lim_{t \rightarrow \infty} \left(\frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \right)$

$$- \frac{t^2+1}{t^2-1} \xrightarrow[t \rightarrow \infty]{} \infty$$

6. $\lim_{t \rightarrow \infty} \left\langle te^{-t}, \frac{t^3 + t}{2t^3 - 1}, t \sin \frac{1}{t} \right\rangle$

7-14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$

$$\mathbf{9.} \quad \mathbf{r}(t) = \langle t, 2 - t, 2t \rangle$$

$$\mathbf{10.} \quad \mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$$

$$\mathbf{11.} \quad \mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$$

12. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \mathbf{k}$

 33–37 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

33. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

$$\mathbf{34.} \quad \mathbf{r}(t) = \langle te^t, e^{-t}, t \rangle$$

$$\mathbf{35.} \quad \mathbf{r}(t) = \langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \rangle$$

36. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$

37. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$