§12.6 Quadric Surfaces

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Wow! I never would've thought to word it that way, but what a neat definition!

$$z = x^2.$$

Notice that the equation of the graph, $z = x^2$, doesn't involve y.

This means that any vertical plane with equation $y = k$(parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$.

So these vertical traces are parabolas.

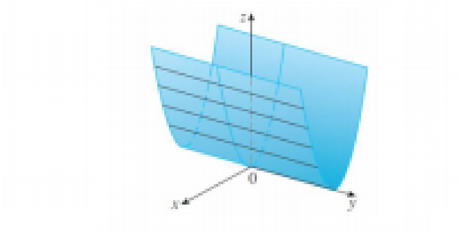
Missing variable is *typical* of cylinders, in general.

But if the rulings are skew with respect to coordinate axes, things can get pretty ugly.

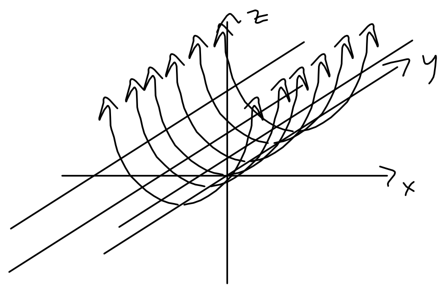
A line to which all rulings are parallel is the y-axis. Or any line parallel to the y-axis.

If you hold the y-value constant, say, fix $y = 0$, you're taking a cross section of this cylinder (actually, we'd interpret it as a "trough," whose sides make a parabola shape.

Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis.



The surface $z = x^2$ is a parabolic cylinder.



Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

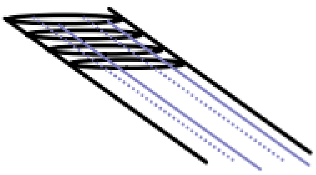
$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

$x^2 + y^2 = 1$ isn't a circle in 3-D. It's a cylinder. One that happens to be

a tube, which was probably our preconception of what a cylinder is.

By our definition, $y = \sin x$ is a cylinder! Imagine corrugated metal siding, stretching up and down, forever.

As for how to handle when these are tilted on their sides, holding one variable constant still gives you a curve, and you're forming traces (slices) that may stack up, like a leaning Tower of Pisa.



Pick a z-value. Draw the circle/ellipse. Drop down the z-value. Draw the circle. Ideally, your cross-sections are normal to the rulings. But this one, we didn't.

We built it so the tracings were parallel to the xy-plane.

EXAMPLE 3 Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

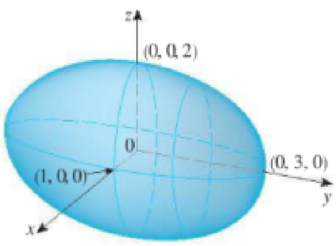


FIGURE 4

The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

$z = 0$ $\quad x^2 + y^2/9 = 1.$

$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z = k$

$k^2 < 4,$

$-2 < k < 2.$

EXAMPLE 4 Use traces to sketch the surface $z = 4x^2 + y^2$.

$x = k$ $\qquad z = y^2 + 4k^2$

are traces in the plane $x = k$, that is, in a plane parallel to the yz-plane. $x = 0$ is the easiest to get...

I imagine there's a cut-off, where if you tilt your traces enough, they would go from ellipses to parabolas, or vice-versa. You think?

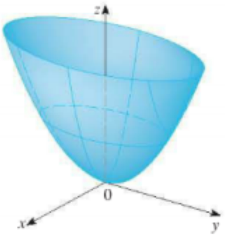
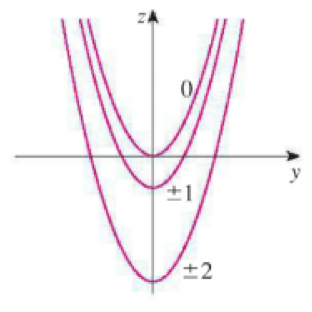


FIGURE 5
The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

I imagine if the coefficients of the x^2 and y^2 were the same, it'd be called a 'circular paraboloid.'

EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

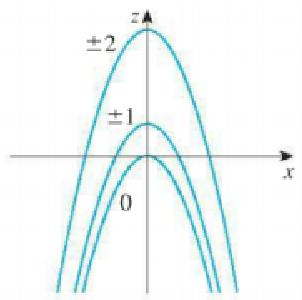


Traces in $x = k$ are $z = y^2 - k^2$

Looking towards the origin, from somewhere on the *x*-axis.

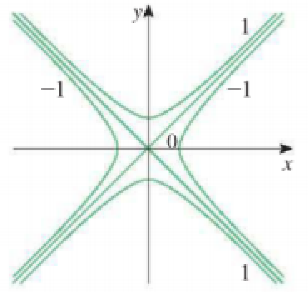
The lowest one is the closest one to us, apparently corresponding to *x* = +/- 2. So we're actually looking 2 of those cross-sections, one behind the other, in this view.

So we're in a valley that cuts into a mountain, and the *y*-axis represents the height of the mountain pass, and there's an identical valley, down the other side of the pass...



Traces in $y = k$ are $z = -x^2 + k^2$

This wasn't much help to *me*, because I couldn't *see* how they tied in with the slices in the previous picture.



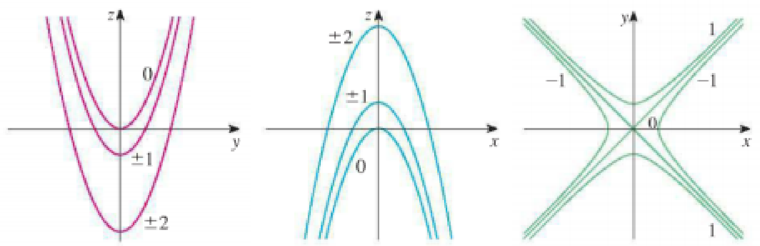
Traces in $z = k$ are $y^2 - x^2 = k$

Oh! NOW I can see it!

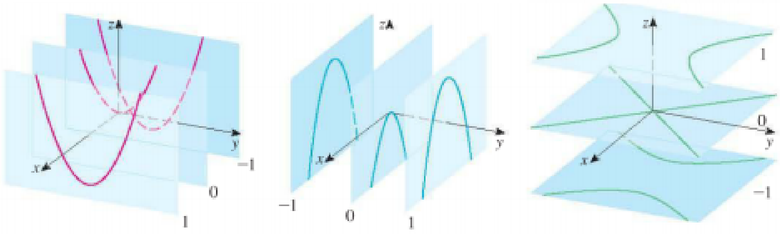
It's a saddle! If I'm sitting on the screen, and my eyes are pointed in the positive *y*-direction, my legs go thru the screen, to either side, and looking ahead, it's rising in front of me and behind me.

And I'm slouching.

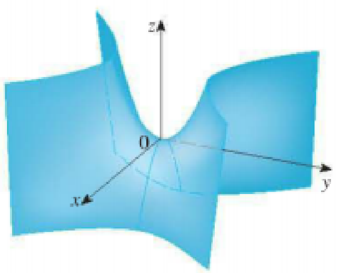
No surprise!



Traces in $x = k$ are $z = y^2 - k^2$ Traces in $y = k$ are $z = -x^2 + k^2$ Traces in $z = k$ are $y^2 - x^2 = k$

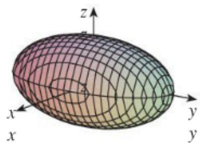
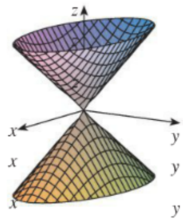
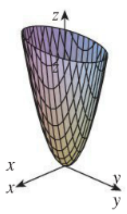
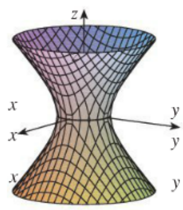
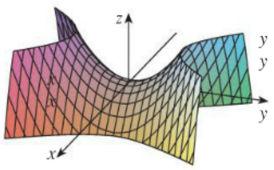
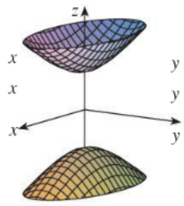


Traces in $x = k$ Traces in $y = k$ Traces in $z = k$



Click on the Earth to see a learning tool from the publisher!

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

1. (a) What does the equation $y = x^2$ represent as a curve in $\mathbb{R}^2$?
(b) What does it represent as a surface in $\mathbb{R}^3$?
(c) What does the equation $z = y^2$ represent?

2. (a) Sketch the graph of $y = e^x$ as a curve in $\mathbb{R}^2$.
(b) Sketch the graph of $y = e^x$ as a surface in $\mathbb{R}^3$.
(c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

4. $4x^2 + y^2 = 4$

9. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.

(b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?

(c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

11–20 Use traces to sketch and identify the surface.

11. $x = y^2 + 4z^2$

13. $x^2 = 4y^2 + z^2$

15. $9y^2 + 4z^2 = x^2 + 36$

13. $x^2 = 4y^2 + z^2$

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

24. $-x^2 + y^2 - z^2 = 1$

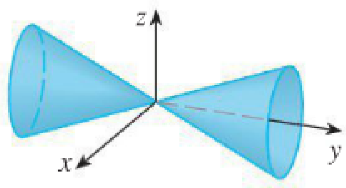
25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

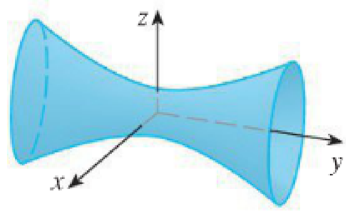
27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

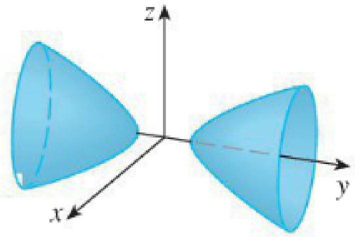
I



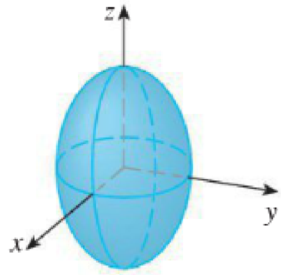
II



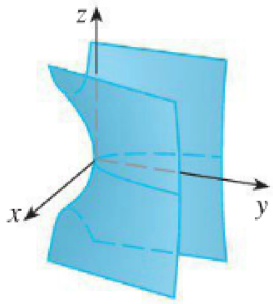
III



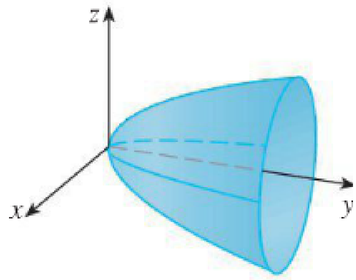
IV



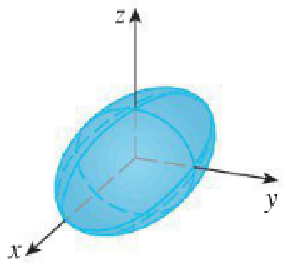
V



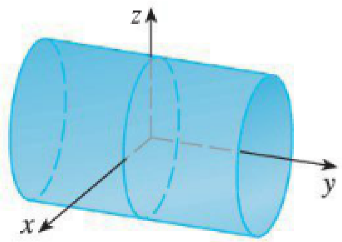
VI



VII



VIII



31–38 Reduce the equation to one of the standard forms, classify the surface, and sketch it. §12.6 #s 1, 2, 4, 9, 11, 13, 15, 21-28, 31, 37

31. $y^2 = x^2 + \frac{1}{9}z^2$

37. $x^2 - y^2 + z^2 - 4x - 2z = 0$