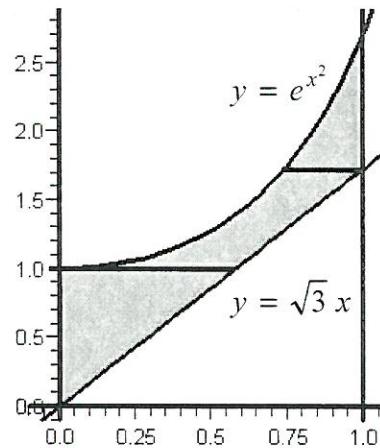


The take-home problems are due Tuesday, at the beginning of class. You may choose to omit any of #s 1 – 6 and receive half-credit for turning them in with the take-home portion. When you do that, you’re betting I’d give you less than half-credit for in-class work.

1. (15 pts) Evaluate the iterated integral $\int_0^1 \int_{\sqrt{3}x}^{e^{x^2}} 8xy \, dy \, dx$. A

sketch of the Type I region R over which this integral is taken is given on the right, with some additional information you might find helpful for #2.



2. (15 pts) Re-write the integral in #1, to give you the volume under $f(x, y) = 4xy$, by viewing R as a Type II region. This will require 3 different iterated integrals. The extra horizontal lines are meant to be a hint. *Do not evaluate!!!*

3. (15 pts) Evaluate the iterated integral $\iint_D (x + y) \, dA$, where D is the region in the 1st quadrant, between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$, by converting to polar coordinates.

4. (15 pts) Evaluate the triple integral $\iiint_{\mathcal{E}} xy \, dV$, where \mathcal{E} is the solid in the first octant bounded by the parabolic cylinders $y = x^2$, $x = y^2$ and the planes $z = 0$ and $z = x + y$

5. (15 pts) Evaluate the triple integral $\iiint_{\mathcal{E}} (x^3 + xy^2) \, dV$, where \mathcal{E} is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$. Hint: Converting to Cylindrical Coordinates after you’ve found the triple integrals limits of integration will make evaluation easier.

6. (15 pts) Compute the Jacobian for the transformation $u = x + y, v = 2x + 3y$.

Take-Home:

7. (5 pts) Use spherical coordinates to evaluate

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} dz dx dy$$

8. (5 pts) Give five other iterated integrals that are equal to $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx$

9. (5 pts) Evaluate the integral in two ways:

- a. as written
- b. using polar coordinates

$$\int_0^1 \int_1^2 \frac{x}{x^2 + y^2} dx dy$$

$$\textcircled{1} \quad \int_0^1 \int_{\sqrt{3}x}^{e^{x^2}} 8xy \, dy \, dx = \int_0^1 \left[8x \cdot \frac{y^2}{2} \right]_{\sqrt{3}x}^{e^{x^2}} \, dx$$

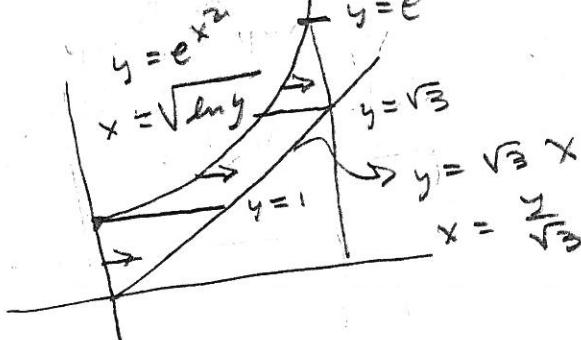
$$= \int_0^1 4x \left[(e^{x^2})^2 - (\sqrt{3}x)^2 \right] \, dx$$

$$= \int_0^1 [4x e^{2x^2} - 4x \cdot 3x^2] \, dx$$

$$= \int_0^1 e^{2x^2} \cdot 4x \, dx - \int_0^1 12x^3 \, dx$$

$$= \left. e^{2x^2} \right|_0^1 - \left. \frac{12}{4} x^4 \right|_0^1 = \frac{e^2 - e^0 - 3}{e^2 - 4}$$

(2)



$$\begin{aligned} y &= e^{x^2} \\ \ln y &= x^2 \\ \pm \sqrt{\ln y} &= x \\ \text{take the } +! \end{aligned}$$

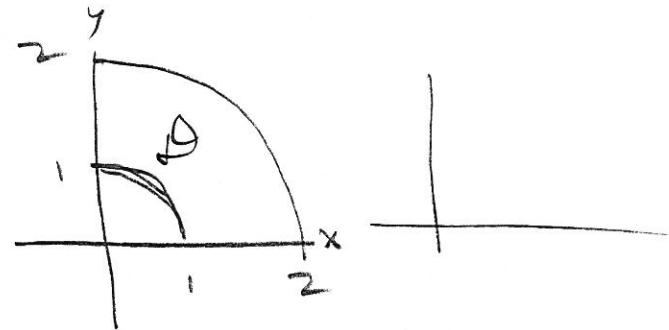
$$\int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{\sqrt{3}}} 8xy \, dx \, dy + \int_{y=1}^{y=\sqrt{3}} \int_{x=\sqrt{\ln y}}^{x=\frac{y}{\sqrt{3}}} 8xy \, dx \, dy$$

$$+ \int_{y=\sqrt{3}}^{y=e} \int_{x=\sqrt{\ln y}}^{x=1} 8xy \, dx \, dy$$

$$\textcircled{3} \quad \iint_D (x+y) dA$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$D = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$$

$$\int_0^{\frac{\pi}{2}} \int_1^2 (r \cos \theta + r \sin \theta) r dr d\theta = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}\}$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 r^2 (\cos \theta + \sin \theta) dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} (\cos \theta + \sin \theta) \right]_1^2 d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{8}{3} - \frac{1}{3} \right) (\cos \theta + \sin \theta) d\theta$$

$$= \frac{7}{3} \int_0^{\frac{\pi}{2}} (\cos \theta + \sin \theta) d\theta = \frac{7}{3} \left[\sin \theta - \cos \theta \right]_0^{\frac{\pi}{2}}$$

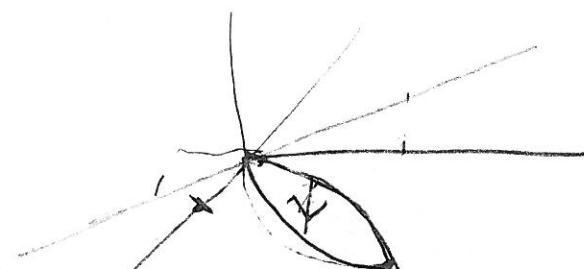
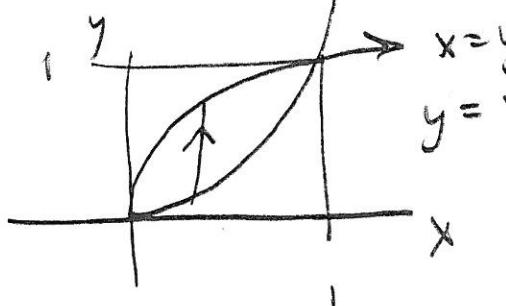
$$= \frac{7}{3} \left[\sin \frac{\pi}{2} - \cos \frac{\pi}{2} - (\sin 0 - \cos 0) \right]$$

$$= \frac{7}{3} [1 - 0 - (0 - 1)] = \boxed{\frac{14}{3}}$$

203 TEST 3

④ $\iiint_{\Sigma} xy \, dV$, Σ is solid bold by

Σ $y = x^2 \Rightarrow x = \sqrt{y}$ $y = x^2$, $x = y^2$ and the planes $z = 0$ & $z = x + y$



$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} \int_{z=0}^{z=x+y} xy \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[xyz \right]_{z=0}^{z=x+y} \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [xy(x+y)] \, dy \, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + xy^2) \, dy \, dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^{y=\sqrt{x}} \, dx = \int_0^1 \left[\frac{x^2\sqrt{x}^2}{2} + \frac{x\cdot\sqrt{x}}{3} \right]^3 \, dx$$

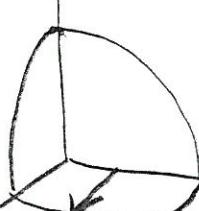
$$= - \left(\frac{x^2(x^2)^2}{2} + \frac{x(x^2)^3}{3} \right) \, dx = \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right] \, dx$$

$$= \left[\frac{x^4}{8} + \frac{2x^{7/2}}{21} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24} = \boxed{\frac{3}{28}}$$

(3)

$\iiint_{E} (x^3 + xy^2) dV$, ant, beneath $z = 1 - x^2 - y^2$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (x^3 + xy^2) dz dy dx$$



convert to cylindrical coordinates

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^2 \cos \theta (r^2 \cos^2 \theta + r^2 \sin^2 \theta) dz dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^4 \cos \theta dz dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos \theta (1-r^2) dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 (r^4 - r^6) \cos \theta dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{5} r^5 - \frac{1}{7} r^7 \right]_0^1 \cos \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{2}{35} \cos \theta d\theta$$

$$= \frac{2}{35} [\sin \theta]_0^{\frac{\pi}{2}} = \boxed{\frac{2}{35}}$$

(b)

$$u = x + y$$

$$v = 2x + 3y$$

$$x + y = u$$

$$2x + 3y = v$$

$$\begin{bmatrix} 1 & 1 & | & u \\ 2 & 3 & | & v \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & u \\ 0 & 1 & | & v - 2u \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & u - v + 2u \\ 0 & 1 & | & v - 2u \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 3u - v \\ 0 & 1 & | & v - 2u \end{bmatrix}$$

$$x = 3u - v$$

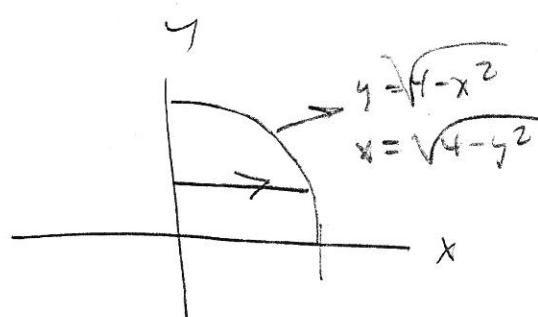
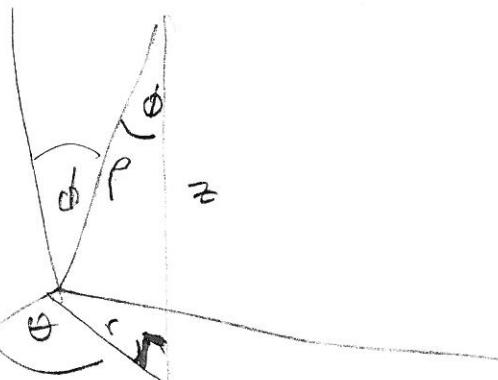
$$y = v - 2u$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} = 3 - 2 = \boxed{1}$$

203 TEST 3

(7)

$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} dz dx dy$$



$$\begin{aligned} x &= \rho \sin \theta \cos \phi \\ y &= \rho \sin \theta \sin \phi \\ z &= \rho \cos \theta \end{aligned}$$

$$dV =$$

$$\begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} =$$

$$\begin{vmatrix} \sin \theta \cos \phi & -\rho \sin \theta \sin \phi \\ \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\ \sin \theta \sin \phi & \rho \cos \theta \sin \phi \\ \cos \theta & -\rho \sin \phi \end{vmatrix}$$

$$= \begin{vmatrix} + & -\rho \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\ + & \rho \sin \theta \cos \phi & \rho \cos \theta \sin \phi \\ - & 0 & -\rho \sin \phi \end{vmatrix}$$

$$= \cancel{\rho \cos \theta \cos \phi} (-\rho \sin \theta \sin \phi - \cancel{\rho \sin \theta \cos \phi \cos \phi})$$

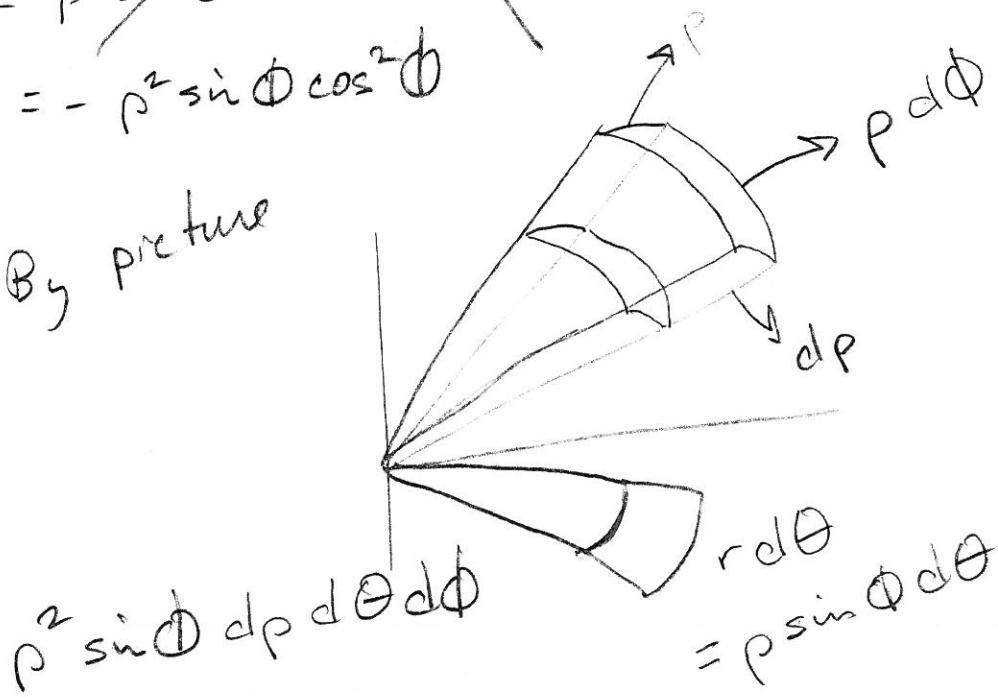
$$- \cancel{\rho \cos \theta \sin \phi} (-\rho \sin^2 \theta \cos \phi + \cancel{\rho \sin \theta \cos \phi \sin \phi})$$

=

203 TEST 3

$$\begin{aligned}
 &= \cancel{\rho^2 \sin \phi \cos \theta \cos \phi (-\sin \phi \sin \theta - \cos \phi \cos \theta)} \\
 &\quad - \cancel{\rho^2 \sin \phi \cos \theta \sin \theta (-\sin \phi \cos \theta + \cos \phi \sin \theta)} \\
 &= \cancel{\rho^2 \sin \phi \cos \phi} \left(\cancel{-\sin \phi \sin \theta \cos \theta - \cos \phi \cos^2 \theta} \right. \\
 &\quad \left. + \cancel{\sin^2 \phi \sin \theta \cos \theta - \cos \phi \sin^2 \theta} \right) \\
 &= \cancel{\rho^2 \sin \phi \cos \phi} \left(-\cos \phi (\cos^2 \theta + \sin^2 \theta) \right) \\
 &= \cancel{\rho^2 \sin \phi \cos \phi} (-\cos \phi) \\
 &= -\rho^2 \sin \phi \cos^2 \phi
 \end{aligned}$$

By picture



203 TEST 3

$$\frac{\mathbf{J}(x, y, z)}{z(p, \theta, \phi)} =$$

$$\begin{aligned}
 &= \cos\phi (-p^2 \sin\phi \cos\phi \sin^2\theta - p^2 \sin\phi \cos\phi \cos^2\theta) \\
 &\quad + -p \sin\phi (p \sin^2\phi \cos^2\theta + p \sin^2\phi \sin^2\theta) \\
 &= -p^2 \cos\phi (\sin\phi \cos\phi (\sin^2\theta + \cos^2\theta)) \\
 &\quad + -p^2 \sin\phi (\sin^2\phi (\cos^2\theta + \sin^2\theta)) \\
 &= -p^2 \cos\phi \sin\phi \cos\phi - p^2 \sin\phi \sin^2\theta \\
 &= -p^2 \sin\phi (\cos^2\phi + \sin^2\phi) \\
 &= -p^2 \sin\phi
 \end{aligned}$$

Apparently, we're integrating over the hemisphere where $x \geq 0$ and $x^2 + y^2 + z^2 = 4$ intersect, so

$$R = \{(p, \theta, \phi) \mid 0 \leq p \leq 2, 0 \leq \theta \leq \pi, 0 \leq \phi \leq \pi\}. \text{ This gives}$$

$$\int_0^\pi \int_0^\pi \int_0^2 (p \sin\phi \sin\theta)^2 \cdot p \cdot p^2 \sin\phi \, dp \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^\pi \int_0^2 p^5 \sin^3\phi \sin^2\theta \, dp \, d\theta \, d\phi$$

203 Test 3

#7 error: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is the error.

$$dV = \rho^2 \sin\phi d\rho d\theta d\phi$$

$$y^2 = \rho^2 \sin^2\phi \sin^2\theta$$

$$\sqrt{x^2 + y^2 + z^2} = \rho$$

$$\int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin^2\phi \sin^2\theta \cdot \rho \cdot \rho^2 \sin\phi d\rho d\theta d\phi$$

$$= \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \rho^5 \sin^3\phi \sin^2\theta d\rho d\theta d\phi$$

$$= \int_0^\pi \sin^3\phi d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta d\theta \int_0^2 \rho^5 d\rho$$

$$= \int_0^\pi (\sin\phi + \cos^2\phi(-\sin\phi)) d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2\theta)) d\theta \left[\frac{2^6}{6} - 0 \right]$$

$$= \left[-\cos\phi + \frac{1}{3}\cos^3\phi \right]_0^\pi \left(\frac{1}{2} \right) \left[\theta - \frac{1}{2}\sin(2\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{64}{6} \right)$$

$$= \left[-(-1) + \frac{1}{3}(-1)^3 - \left(+1 + \frac{1}{3} \right) \right] \left(\frac{1}{2} \right) \left[\frac{\pi}{2} - \theta - \left(-\frac{\pi}{2} - \theta \right) \right] \left(\frac{32}{3} \right)$$

$$= \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right] \left(\frac{1}{2} \right) [\pi] \left(\frac{32}{3} \right) = \boxed{\frac{64\pi}{9}}$$

Extra Work on #7

$$\int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 p^5 \sin^3 \phi \sin^2 \theta \, dp \, d\theta \, d\phi$$

$$= \int_0^\pi \sin^3 \phi \, d\phi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \int_0^2 p^5 \, dp = I_1 I_2 I_3 \Rightarrow$$

$$I_1 = \int_0^\pi \sin \phi (1 - \cos^2 \phi) \, d\phi = \int_0^\pi \sin \phi \, d\phi + \int_0^\pi \cos^2 \phi (-\sin \phi) \, d\phi$$

$$= -\cos \phi \Big|_0^\pi + \frac{1}{3} \cos^3 \phi \Big|_0^\pi = 1 - (-1) + \frac{1}{3}(-1) - \frac{1}{3}(1)$$

$$= 2 - \frac{2}{3} = \boxed{\frac{4}{3}} = I_1$$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta \, d\theta = 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos(2\theta)) \, d\theta = \theta \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \sin(2\theta) \Big|_0^{\frac{\pi}{2}}$$

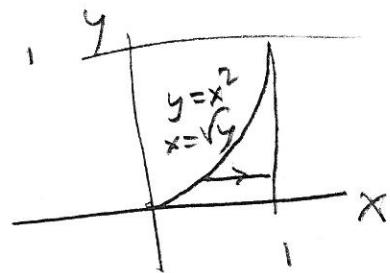
$$= \boxed{\frac{\pi}{2}} = I_2$$

$$I_3 = \int_0^2 p^5 \, dp = \left[\frac{p^6}{6} \right]_0^2 = \frac{64}{6} = \boxed{\frac{32}{3}} = I_3$$

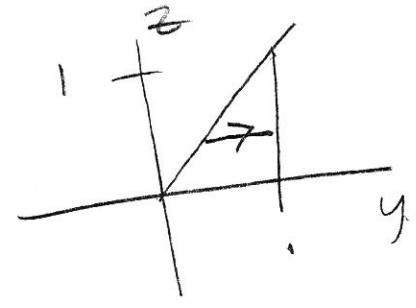
$$\Rightarrow I_1 I_2 I_3 = \boxed{\frac{64\pi}{9}}$$

(8)

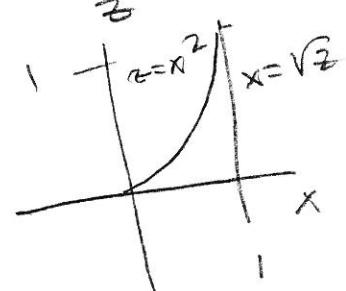
$$\int_0^1 \int_0^{x^2} \int_0^y f \, dz \, dy \, dx$$



$$= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f \, dz \, dx \, dy$$

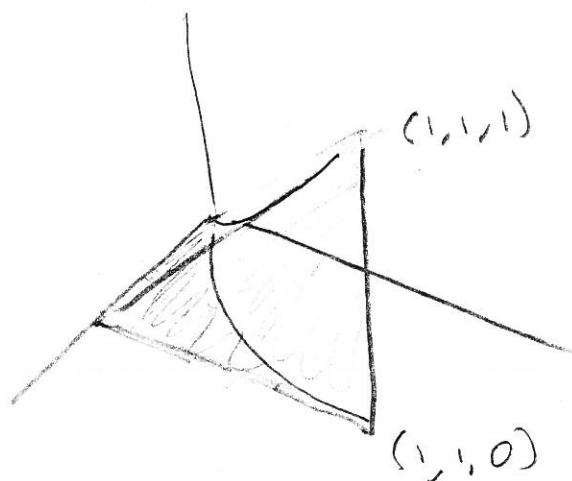


$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f \, dx \, dz \, dy$$

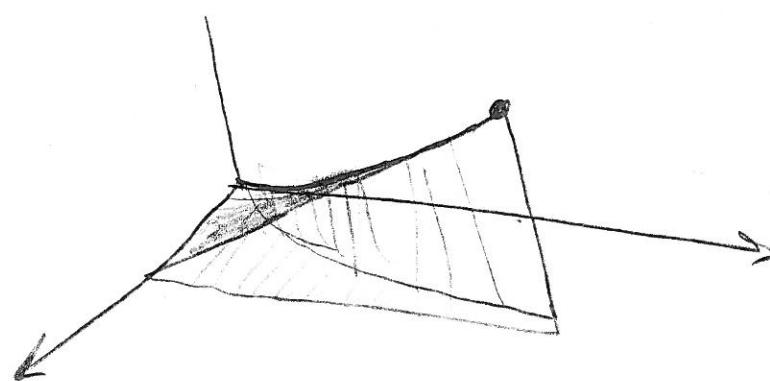


$$= \int_0^1 \int_0^1 \int_{\sqrt{y}}^1 f \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^{x^2} \int_x^{x^2} f \, dy \, dz \, dx$$

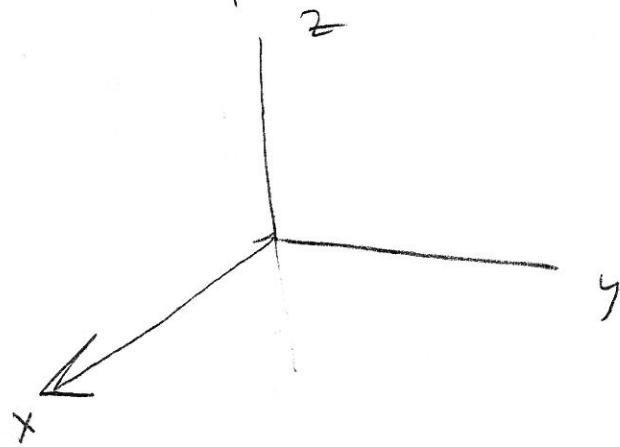
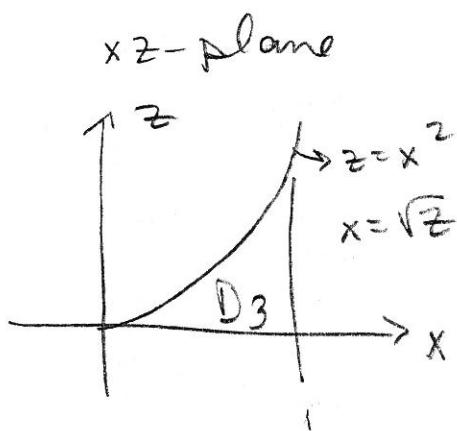
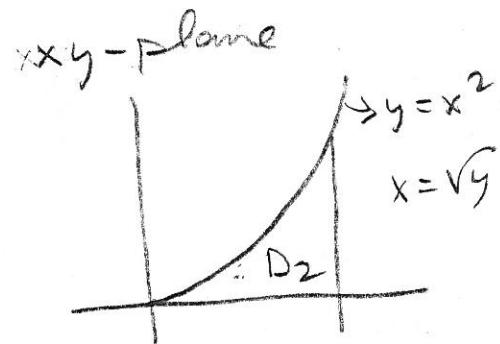
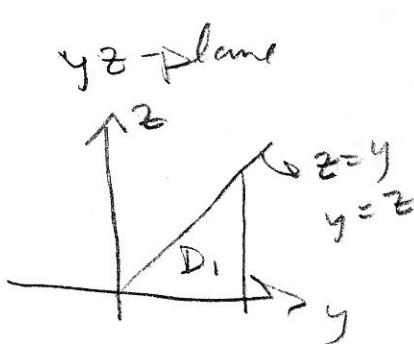


$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f \, dy \, dx \, dz$$



$$\int_0^1 \int_0^{x^2} \int_0^y f \, dz \, dy \, dx$$

$$E = \{(x, y, z) \mid 0 \leq z \leq y, 0 \leq y \leq x^2, 0 \leq x \leq 1\}$$



$$D_2 = \{(x, y) \mid 0 \leq y \leq x^2, 0 \leq x \leq 1\}$$

$$= \{(x, y) \mid \sqrt{y} \leq x \leq 1, 0 \leq y \leq 1\}$$

$$D_1 = \{(y, z) \mid 0 \leq z \leq y, 0 \leq y \leq 1\}$$

$$= \{(y, z) \mid 0 \leq z \leq 1, z \leq y \leq 1\}$$

$$D_3 = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x^2\}$$

$$= \{(x, z) \mid \sqrt{z} \leq x \leq 1, 0 \leq z \leq 1\}$$

203 Test 3

$$\textcircled{9} \int_0^1 \int_1^2 \frac{x}{x^2+y^2} dx dy$$

$$= \frac{1}{2} \int_0^1 \int_1^2 \frac{2x}{x^2+y^2} dx dy$$

$$= \frac{1}{2} \int_0^1 \left[\ln(x^2+y^2) \right]_1^2 dy$$

$$= \frac{1}{2} \int_0^1 (\ln(y^2+4) - \ln(y^2+1)) dy$$

$$= \frac{1}{2} \int_0^1 \ln(y^2+4) dy - \frac{1}{2} \int_0^1 \ln(y^2+1) dy$$

$$= \frac{1}{2}(I_1 - I_2)$$

$$I_1 = \int_0^1 \ln(y^2+4) dy$$

$$= y \ln(y^2+4) \Big|_0^1 - \int_0^1 y \cdot \frac{2y}{y^2+4} dy$$

$$= \ln(5) - 2 \int_0^1 \left(1 - \frac{4}{y^2+4}\right) dy$$

$$= \ln 5 - \int_0^1 dy + 4 \int_0^1 \frac{1}{y^2+4} dy$$

$$= \ln 5 - 2 + 8 \cdot \frac{1}{2} \arctan\left(\frac{y}{2}\right) \Big|_0^1$$

$$= \boxed{\ln 5 - 2 + 4 \arctan\left(\frac{1}{2}\right)} = I_1$$

$$u = \ln(y^2+4) \quad dv = dy \\ du = \frac{2y}{y^2+4} \quad v = y$$

$$y^2+4 \quad \cancel{y^2+4y+0} \\ - \underline{(y^2+4)}$$

203 Test 3.

⑨ $I_2 = \int_0^1 \ln(y^2+1) dy$ Similar to I_1 ,
only y^2+1 instead of
 y^2+4

$$= y \ln(y^2+1) \Big|_0^1 - \int_0^1 \frac{2y^2}{y^2+1} dy$$
$$= \ln 2 - 2 \int_0^1 \left(1 - \frac{1}{y^2+1}\right) dy$$
$$= \ln 2 - 2 \left[y - \arctan(y)\right]_0^1$$
$$= \ln 2 - 2 \left[1 - \frac{\pi}{4}\right]$$
$$\boxed{\ln 2 - 2 + \frac{\pi}{2} = I_2}$$

$$0^o \quad \frac{1}{2}[I_1 - I_2] = \frac{1}{2} \left[\ln 5 - 2 + 4 \arctan\left(\frac{1}{2}\right) \right. \\ \left. - \ln 2 + 2 - \frac{\pi}{2} \right]$$
$$= \frac{1}{2} \left[\ln 5 - \ln 2 + 4 \arctan\left(\frac{1}{2}\right) - \frac{\pi}{2} \right]$$
$$= \frac{1}{2} \ln\left(\frac{5}{2}\right) + 2 \arctan\left(\frac{1}{2}\right) - \frac{\pi}{4}$$
$$\boxed{= \frac{1}{2} \ln\left(\frac{5}{2}\right) + 2 \arctan\left(\frac{1}{2}\right) - \frac{\pi}{4}}$$