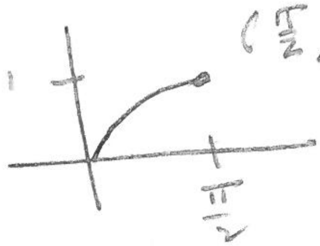
203 TAKE-HOME Spring 2019

1 10 pts

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{\cos^2 x+1}\, dx\, dy$$

$x = \arcsin y \longrightarrow y = \sin x$!

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin x} * \, dy\, dx$$



$y = \sin(x)$

$x = \arcsin(y)$

$$= \int_0^{\frac{\pi}{2}} \left[\left(\cos(x)\sqrt{\cos^2 x + 1}\right) y \right]_{y=0}^{\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos(x)\sqrt{\cos^2 x + 1}\, \sin(x)\, dx$$

$u = \cos x \longrightarrow du = -\sin(x)\, dx$

$$= -\int_{x=0}^{x=\frac{\pi}{2}} u\sqrt{u^2+1}\, du$$

Let $v = u^2+1 \longrightarrow dv = 2u\, du$

$$= -\frac{1}{2}\int_{x=0}^{x=\frac{\pi}{2}} \sqrt{u^2+1}\cdot 2u\, du = -\frac{1}{2}\int_{0=x}^{x=\frac{\pi}{2}} v^{\frac{1}{2}}\, dv$$

$$= -\left(\frac{1}{2}\right)\frac{2}{3}\, v^{\frac{3}{2}} \Big]_{x=0}^{x=\frac{\pi}{2}} = -\frac{1}{3}\left(u^2+1\right)^{\frac{3}{2}} \Big]_{x=0}^{x=\frac{\pi}{2}}$$

$$= -\frac{1}{3}\left(\cos^2 x + 1\right)^{\frac{3}{2}} \Big]_{x=0}^{x=\frac{\pi}{2}} = -\frac{1}{3}\left[\left(0^2+1\right)^{\frac{3}{2}} - (1+1)^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}\left[1 - 2\sqrt{2}\right) = \boxed{\frac{2\sqrt{2}-1}{3}}$$

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2 a 10pts $\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle$

$\Longrightarrow \bar{r}_r = \langle \cos\theta, \sin\theta \rangle$

& $\bar{r}_\theta = \langle -r\sin\theta, r\cos\theta \rangle$

$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \left| \bar{r}_r \times \bar{r}_\theta \right|$$

$\langle \cos\theta, \sin\theta, 0 \rangle \cos\theta, \sin\theta$

$\times \langle -r\sin\theta, r\cos\theta, 0 \rangle -r\sin\theta, r\cos\theta$

$\langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle = \bar{r}_r \times \bar{r}_\theta$

$\longrightarrow \left| \bar{r}_r \times \bar{r}_\theta \right| = \sqrt{0^2 + 0^2 + (r\cos^2\theta + r\sin^2\theta)^2}$

Alternate $\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$

$= \sqrt{r^2} = \boxed{r}$

2b i 10pts

Between

$z = 3x^2 + 3y^2 = 3r^2$ Lower

& $z = 4 - x^2 - y^2 = 4 - r^2$ upper

$z_1 = z_2$

$3r^2 = 4 - r^2 \Longrightarrow 4r^2 = 4 \Longrightarrow r = 1$

So, $\int_0^{2\pi}\int_0^1 \left((4-r^2) - 3r^2\right) r \, dr \, d\theta = \int_0^{2\pi}\int_0^1 (4-4r^2) \, r \, dr \, d\theta$

$= \int_0^{2\pi}\int_0^1 (4r - 4r^3) \, dr \, d\theta$

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2bi cont'd

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{4r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi} (2-1)\, d\theta = \boxed{2\pi}$$

2bii 10pts

Inside $x^2+y^2+z^2 = 16$ &

Outside $x^2+y^2=4$

$r^2=4 \Rightarrow r = \pm 2$

$$z = \pm\sqrt{16-r^2}$$

outside the cylinder : $r=2$ to $r=4$

upper $= \sqrt{16-r^2}$, lower $= -\sqrt{16-r^2} \longrightarrow$

upper - lower $= 2\sqrt{16-r^2}$

$$\Rightarrow \text{Area} = \int_0^{2\pi}\int_2^4 2\sqrt{16-r^2}\; r\, dr\, d\theta$$

$$= -\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\cdot(-2r\, dr\, d\theta)$$

$u = 16-r^2 \Rightarrow du = -2r\, dr$

$$= -\int_0^{2\pi}\int_{r=2}^{r=4} u^{\frac{1}{2}} du = -\int_0^{2\pi}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{r=2}^{r=4} d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}\left[16-r^2\right]^{\frac{3}{2}}_{r=2}^{r=4} = -\frac{2}{3}\int_0^{2\pi} 0 - 12^{\frac{3}{2}}$$

$$= \frac{2}{3}\int_0^{2\pi} 12\sqrt{12}\, d\theta = 8\int_0^{2\pi} 2\sqrt{3}\, d\theta = 16\sqrt{3}\int_0^{2\pi} d\theta$$

$$= \boxed{32\sqrt{3}\,\pi}$$

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(3) 10 pts $\bar{r} = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle$

$\longrightarrow$

$\bar{r}_\rho = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$,

$\bar{r}_\phi = \langle \rho\cos\phi \cos\theta, \rho\cos\phi \sin\theta, -\rho \sin\phi \rangle$

$\bar{r}_\theta = \langle -\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, 0 \rangle$

$|\bar{r}_\theta \circ (\bar{r}_\rho \times \bar{r}_\phi)| :$

$$\left| \begin{matrix} -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \end{matrix} \right|$$

$= -\rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)$

$\quad - \rho\sin\phi\cos\theta\left(-\rho\sin^2\phi\cos\theta - \rho\cos^2\phi\cos\theta\right)$

$= -\rho\sin\phi\sin\theta\left(-\rho\sin\theta\right)$

$\quad -\rho\sin\phi\cos\theta\left(-\rho\cos\theta\right)$

$= \rho^2\sin\phi\sin^2\theta + \rho^2\sin\phi\cos^2\theta$

$= \rho^2\sin\phi$ |

$= \left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|$

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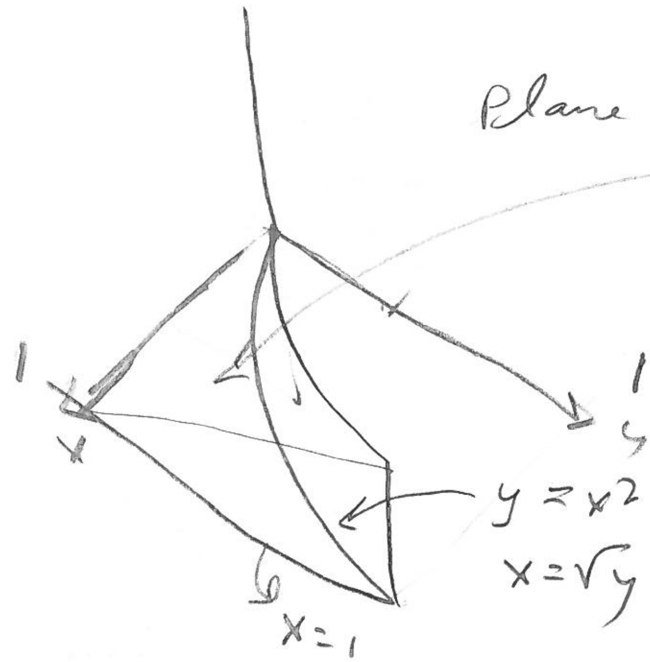
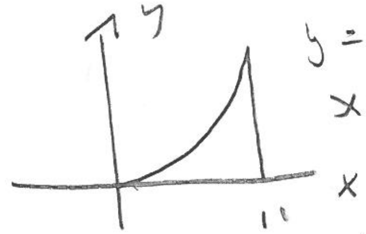
(4) (10 pts) $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$

Plane $z = y = z$



$y = x^2$
$x = \sqrt{y}$



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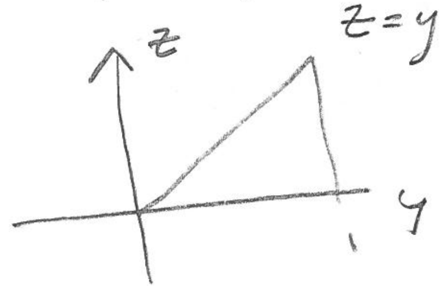
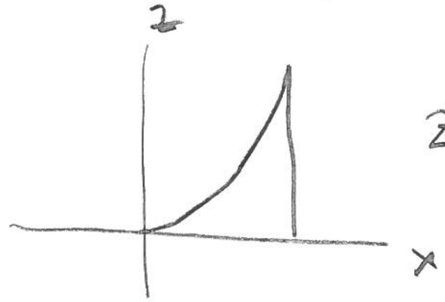


$z = y = z$



$z = x^2$
$x = \sqrt{z}$

via $z = y$ & $y = x^2$

$$= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f\, dz\, dx\, dy = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f\, dy\, dx\, dz$$

$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f\, dy\, dz\, dx = \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f\, dx\, dz\, dy$$

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5

10 pts

$x+y=2, \quad x+y=-2,$

$x+2y-z=4, \quad x+2y-z=-2$

$2x-2y-z=-2, \quad 2x-2y-z=2$

$u=x+y, \quad v=x+2y-z, \quad w=2x-2y-z$

Quick & Dirty. If $T(x,y,z)=(u,v,w)$,

then $\det(T)=\dfrac{1}{\det(T^{-1})}$ & $\det(T^{-1})$

$= \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$!

$$|T| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & -2 & -1 \end{vmatrix} =$$

$= 1(-2-2) - 1(-1+2) + 0(\sim)$

$= -4-1=-5 \longrightarrow \big||T|\big| = 5 \longrightarrow$

$\big||T^{-1}|\big| = \frac{1}{5} = \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| \longrightarrow$

$$\text{Volume} = \frac{1}{5}\int_{-2}^{2}\int_{-2}^{4}\int_{-2}^{2} dw\, dv\, du$$

$= \frac{1}{5}\int_{-2}^{2} du \int_{-2}^{4} dv \int_{-2}^{2} dw$ by Fubini

$= \frac{1}{5}\left(u\Big]_{-2}^{2}\right)\left(v\Big]_{-2}^{4}\right)\left(w\Big]_{-2}^{2}\right) = \dfrac{(4)(6)(4)}{5} = \boxed{\dfrac{96}{5}}$