203 TEST 3 Spring 2019

(1) (10 pts) $\iint_R x^2y\,dA$, $R = [0,3]\times[0,2]$

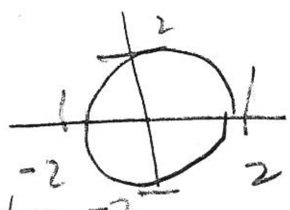
$$= \int_0^2\int_0^3 x^2y\,dx\,dy = \int_0^2 \left[\frac{1}{3} x^3 y\right]_{x=0}^{x=3} dy = \frac{1}{3}\int_0^2 [27y]\,dy$$

$$= \frac{1}{3}\left[\frac{27}{2}y^2\right]_0^2 = 9\left(\frac{2^2}{2}\right) = \boxed{18}$$

$$= \int_0^3\int_0^2 x^2y\,dy\,dx = \int_0^3 x^2\left[\frac{y^2}{2}\right]_0^2 dx = 2\left[\frac{x^3}{3}\right]_0^3 = 2[9] = \boxed{18}$$

(2) (a) (10 pts) $x^2+y^2=4$, $x^2+y^2+z^2=9$

Polar Coordinates.



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^2 f(r,\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (\text{upper} - \text{lower})\,r\,dr\,d\theta$$

$$z = \pm\sqrt{9-x^2+y^2} = \pm\sqrt{9-r^2}$$

upper = +, lower = −

$$= \int_0^{2\pi}\int_0^2 \left(\sqrt{9-r^2} - (-\sqrt{9-r^2})\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 2\sqrt{9-r^2}\,r\,dr\,d\theta = \int_0^2$$

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2a cont'd

$$= -\int_0^{2\pi}\int_0^2 (9-r^2)^{\frac{1}{2}}(-2r)\,dr\,d\theta$$

$u = 9-r^2 \Rightarrow du = -2r\,dr$

$$= -\int_{0=\theta}^{2\pi=\theta}\int_{0=r}^{2=r} u^{\frac{1}{2}}\,du\,d\theta = -\int_0^{2\pi} \frac{2}{3}u^{\frac{3}{2}}\Big]_{r=0}^{r=2} d\theta$$

$$= -\int_0^{2\pi} \frac{2}{3}(9-r^2)^{\frac{3}{2}}\Big]_0^2 d\theta = -\frac{2}{3}\int_0^{2\pi}\left[(9-4)^{\frac{3}{2}} - (9-0)^{\frac{3}{2}}\right]d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}\left(5^{\frac{3}{2}} - 9^{\frac{3}{2}}\right)d\theta = -\frac{2}{3}(5\sqrt{5}-27)\int_0^{2\pi} d\theta$$

$$-\frac{2}{3}(5\sqrt{5}-27)(2\pi) = \frac{108\pi}{3} - \frac{10\sqrt{5}}{3} = \boxed{36\pi - \frac{20\sqrt{5}}{3}\pi}$$

2b 10 pts Cylindrical.

$$\int_0^{2\pi}\int_0^2\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 zr\Big]_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} dr\,d\theta = \underbrace{\int_0^{2\pi}\int_0^r 2\sqrt{9-r^2}\,r\,dr\,d\theta}$$

$$= \cdots = \boxed{\left(36 - \frac{20\sqrt{5}}{3}\right)\pi}$$

by previous work.

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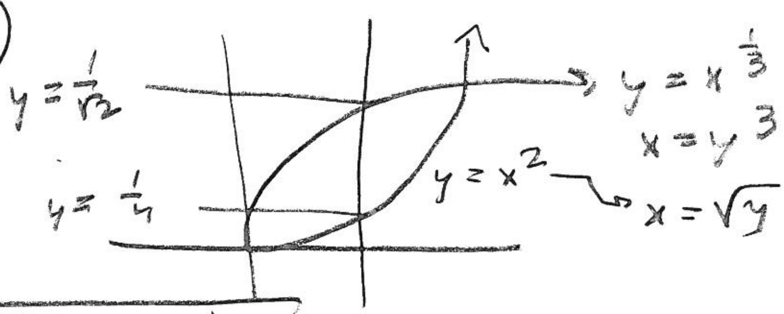
3a   10 pts



$$\boxed{\text{TI: } \int_0^{\frac{1}{2}} \int_{x^2}^{x^{\frac{1}{3}}} x^2 y \, dy \, dx} \quad \text{STOP!}$$

$$= \int_0^{\frac{1}{2}} \left[x^2 \cdot \frac{y^2}{2} \right]_{y=x^2}^{y=x^{\frac{1}{3}}} dx = \int_0^{\frac{1}{2}} \left[\frac{x^2 \cdot x^{\frac{2}{3}}}{2} - \frac{x^2 \cdot x^4}{2} \right] dx$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^{\frac{8}{3}}}{2} - \frac{x^6}{2} \right] dx = \frac{1}{2} \int_0^{\frac{1}{2}} \left(x^{\frac{8}{3}} - x^6 \right) dx$$

$$= \frac{1}{2} \left[\frac{3}{11} x^{\frac{11}{3}} - \frac{1}{7} x^7 \right]_0^{\frac{1}{2}} = \frac{1}{2} \left[\frac{3}{11} \left(\frac{1}{2} \right)^{\frac{11}{3}} - \frac{1}{7} \cdot \left(\frac{1}{2} \right)^7 \right]$$

$$= \frac{1}{2} \cdot \frac{3}{11} \cdot \left(\frac{1}{2} \right)^{\frac{9}{3}} \left(\frac{1}{2} \right)^{\frac{2}{3}} - \frac{1}{2} \left(\frac{1}{7} \right) \left(\frac{1}{2} \right)^7 = \frac{1}{16} \cdot \frac{1}{11} \cdot \frac{2^{\frac{1}{3}}}{2} - \frac{1}{2^8 \cdot 7}$$

$$= \frac{1}{2} \cdot \frac{3}{11} \cdot \frac{1}{2^3} \cdot \frac{1}{(2^2)^{1/3}} - \frac{1}{2^8} \cdot \frac{1}{7}$$

$$= \frac{3\sqrt[3]{2}}{32 \cdot 11} - \frac{1}{1792} = \boxed{\frac{3\sqrt[3]{2}}{352} - \frac{1}{1792}}$$

$$\begin{array}{r} 320 \\ 32 \\ \hline 352 \end{array}$$

$\approx .0101799278$

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3b 10 pts T II See drawing

$$\boxed{\int_0^{\frac14}\int_{y^3}^{\sqrt{y}} x^2y\,dx\,dy + \int_{\frac14}^{\frac{1}{\sqrt[3]{2}}}\int_{y^3}^{\frac12} x^2y\,dx\,dy}$$

$= \frac{3\sqrt[3]{2}}{352} - \frac{1}{1792}$ $\approx \cancel{.0101799}$ $\approx .0101799277$

Stop!

$$= \int_0^{\frac14}\left[\frac13 x^3 y\right]_{y^3}^{y^{\frac12}} dy + \int_{\frac14}^{\frac{1}{\sqrt[3]{2}}}\left[\frac13 x^3 y\right]_{y^3}^{\frac12} dy$$

$$= \int_0^{\frac14}\left[\frac13 (y^{\frac12})^3 y - \frac13 (y^3)^3 y\right] dy + \int_{\frac14}^{\frac{1}{\sqrt[3]{2}}}\left[\frac13 \left(\frac12\right)^3 y - \frac13 (y^3)^3 y\right] dy$$

Ugh! That's why you want to stop!

$$= \frac13 \int_0^{\frac14}\left((y^{3/2} y^1)\,dy - y^9 y\right) dy + \frac13\int_{\frac14}^{\frac{1}{\sqrt[3]{2}}}\left(\frac18 y - y^{10}\right) dy$$

$$= \frac13\left[\int_0^{\frac14}\left(y^{\frac57} - y^{10}\right) dy + \int_{\frac14}^{2^{-\frac13}}\left(\frac18 y - y^{10}\right) dy\right]$$

$$= \frac13\left[\frac{7}{12} y^{\frac{12}{7}} - \frac{1}{11} y^{11}\right]_0^{\frac14} + \frac13\left[\frac1{16} y^2 - \frac1{11} y^{11}\right]_{\frac14}^{2^{-\frac13}}$$

$$= \frac13\left[\left(\frac{7}{12}\left(\frac14\right)\left(\frac14\right)^{\frac57} - \frac1{11}\left(\frac14\right)^{11}\right) + \left(\frac1{16}\cdot\frac{1}{2^{2/3}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} - \frac1{11}\left(\frac{1}{\sqrt[3]{2}}\right)^{11}\right) - \left(\frac1{16}\left(\frac14\right)^2 - \frac1{11}\left(\frac14\right)^2\right)\right]$$

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4 10 pts $I = \iiint_R (9 - x^2 - y^2)\, dV$, where

$R = \{ (x,y,z) \mid x^2+y^2+z^2 \le 9 \;\&\; z \ge 0 \}$

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$= \{ (\rho, \theta, \phi) \mid 0 \le \rho \le 3,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi \}$

$$I = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \left(9 - \rho^2 \sin^2\phi \cos^2\theta - \rho^2 \sin^2\phi \sin^2\theta \right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \left(9 - \rho^2 \sin^2\phi \right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(9\rho^2 \sin\phi - \rho^4 \sin^3\phi \right) d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(9 \cdot \frac{\rho^3}{3} \sin\phi - \frac{1}{5}\rho^5 \underline{\sin^3\phi} \right) \Big]_{0=\rho}^{3=\rho} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(3 \cdot 3^3 \sin\phi - \frac{3^5}{5} \underline{(1 - \cos^2\phi)\sin\phi} \right) d\phi \, d\theta$$

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4 cont'd $= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(3^4 \sin\phi - \frac{3^5}{5}\sin\phi + \frac{3^5}{5}\cos^2\phi \sin\phi\right) d\phi\, d\theta$

$\frac{3^4 \cdot 5}{5} - \frac{3^5}{5} = \frac{3^4(5-3)}{5} = \frac{2 \cdot 3^4}{5}$

$= \frac{2\cdot 3^4}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi\, d\theta + \frac{3^5}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^2\phi \sin\phi\, \cancel{d\phi d}\cdot d\phi\, d\theta$

$= \frac{2\cdot 3^4}{5}\int_0^{2\pi} (-\cos\phi)\Big]_0^{\frac{\pi}{2}} d\theta + \frac{3^5}{5}\int_0^{2\pi} -\frac{\cos^3\phi}{3}\Big]_0^{\frac{\pi}{2}} d\theta$

$= \frac{2\cdot 3^4}{5}\int_0^{2\pi}(-0+1)\, d\theta + \frac{3^5}{5}\int_0^{2\pi}\left(-0+\frac{1}{3}\right) d\theta$

$= \frac{2\cdot 3^4}{5}\left[\theta\right]_0^{2\pi} + \frac{3^5}{5}\left[\frac{1}{3}\theta\right]_0^{2\pi}$

3, 9, 27, 81, 243 — 2, 3, 4, 5

$= \frac{2\cdot 3^4}{5}[2\pi] + \frac{3^4}{5}[2\pi]$

$= \frac{3^4}{5}[4\pi + 2\pi] = \frac{3^4}{5}\cdot 6\pi = \frac{3^5 \cdot 2\pi}{5} = \frac{243\cdot 2\pi}{5}$

$\boxed{= \frac{486\pi}{5}}$

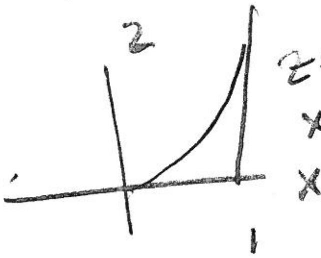
203 T3

B1 10pts

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$$

$z = y = z$

$y = x^2,\ x = \sqrt{y}$

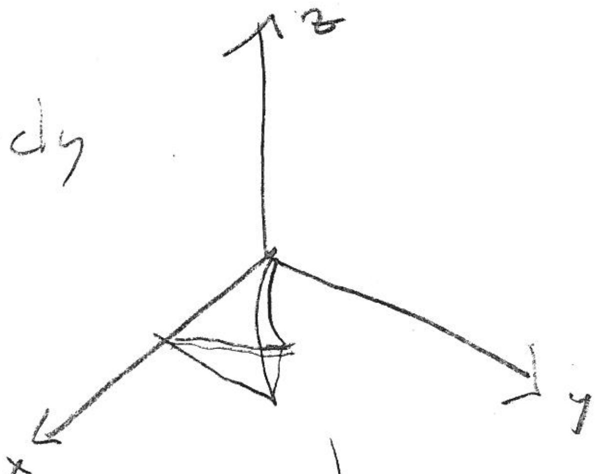


$z = x^2$

$x = \sqrt{z}$ since $z = y = x^2$

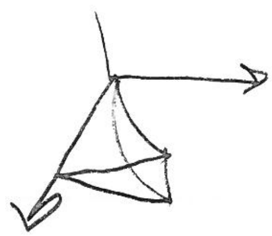
$z = x^2$

$x = \sqrt{z}$



$$= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f\, dz\, dx\, dy$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f\, dy\, dx\, dz$$



$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f\, dy\, dz\, dx$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f\, dx\, dy\, dz = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f\, dx\, dz\, dy$$

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B2 10 pts

$x+y=3=u$ $x+2y-z=6=v$

$x+y=-2=u$ $x+2y-z=1=v$

$w=2x-2y-z=-3$ $2x-2y-z=5=w$

- Quickest & Dirtiest $\bar{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\langle x,y,z\rangle$

$T: xyz \longrightarrow uvw$

$$T\bar{x}=\begin{bmatrix}1&1&0\\1&2&-1\\2&-2&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

FACT $\text{Det}(T)=\dfrac{1}{\text{Det}(T^{-1})}$

$\left\|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right\|=\left|\dfrac{1}{\text{Det}(T)}\right|$, so, compute $\det(T)$

$$\left|1(-2-2)-1(-1+2)\right|=\left|-4-1(1)\right|=\left|-5\right|=5$$

$$\Longrightarrow \left|\bar{r}_u\cdot(\bar{r}_v\times\bar{r}_w)\right|=\frac{1}{5}\ !$$

Hard way?

$$\left[\begin{array}{ccc|c}1&1&0&u\\1&2&-1&v\\2&-2&-1&w\end{array}\right]\begin{array}{l}R1\\-R1+R2\\-2R1+R3\end{array}\left[\begin{array}{ccc|c}1&1&0&u\\0&1&-1&-u+v\\0&-4&-1&-2u+w\end{array}\right]$$

$$\begin{array}{l}-R2+R1\\R2\\4R2+R3\end{array}\left[\begin{array}{ccc|c}1&0&1&2u-v\\0&1&-1&-u+v\\0&0&-5&-6u+4v+w\end{array}\right]$$

R3 $\boxed{z=\frac{1}{5}(6u+4v+w)}$

R2 $y-z=y-\frac{6}{5}u-\frac{4}{5}v-\frac{1}{5}w=-u+v$

$\boxed{y=\frac{1}{5}u+\frac{9}{5}v+\frac{1}{5}w}$

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$\boxed{\text{B2 Cntd}}$

$$x + z = x + \frac{6}{5}u + \frac{4}{5}v + \frac{1}{5}w = 2u - v$$

$$\boxed{x = \frac{4}{5}u - \frac{9}{5}v - \frac{1}{5}w}$$

So, $\bar{r}(u,v,w) = \frac{1}{5}\langle 4u - 9v - w,$

$$= \frac{1}{5}\langle 4u - 9v - w,\ u + 9v + w,\ 6u + 4v + w \rangle$$

$$\rightarrow \bar{r}_u = \frac{1}{5}\langle 4, 1, 6 \rangle$$

$$\bar{r}_v = \frac{1}{5}\langle -9, 9, 4 \rangle \quad \Longrightarrow$$

$$\bar{r}_w = \frac{1}{5}\langle -1, 1, 1 \rangle$$

$$\left| \bar{r}_u \cdot (\bar{r}_v \times \bar{r}_w) \right| = \frac{1}{5^3} \left\| \begin{vmatrix} 4 & 1 & 6 \\ -9 & 9 & 4 \\ -1 & 1 & 1 \end{vmatrix} \right\|$$

$$= \frac{1}{5^3} \left| 4(9-4) - 1(-9+4) + 6(-9+9) \right|$$

$$= \frac{1}{5^3} \left| 4(5) - (-5) + 0 \right| = \frac{1}{5^3} |25| = \frac{1}{5}\ !$$

Now $$\int_{-2}^{3} \int_{1}^{6} \int_{-3}^{5} \left(\frac{1}{5}\right) dw\, dv\, du$$

$$= \frac{1}{5}\left(u\Big]_{-2}^{3}\right)\left(v\Big]_{1}^{6}\right)\left(w\Big]_{-3}^{5}\right)$$

$$= \frac{1}{5}(3-(-2))(6-1)(5-(-3)) = \frac{1}{5}(5)(5)(8)$$

$$\boxed{= 40!}$$