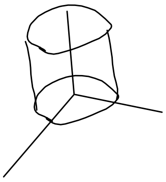
41. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 1$.

$$\int_{S'} \bar{F}\cdot d\bar{S} = \int_{S'} \bar{F}\cdot\bar{n}\,dS = \int_{S'} \bar{F}\cdot(\bar{r}_u \times \bar{r}_v)\,dA$$

$\bar{v} = \langle z, y^2, x^2\rangle$, $x,y,z \sim m$, $\bar{v} \sim m/s$, $S' = \{(x,y,z) \mid x^2+y^2=4,\ 0\le z\le 1\}$



$S_1' =$ Bottom, $S_2' =$ Top, $S_3' =$ sides

Bottom : $\bar{n}_1 = \langle 0,0,-1\rangle$ (But lets work it out like a machine.

$0 \le r \le 2$, $0 \le \theta \le 2\pi$, $z = 0$!

$$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$$
$$\bar{r}_r = \langle \cos\theta, \sin\theta, 0\rangle, \ \cos\theta, \sin\theta$$
$$\times\ \bar{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle - r\sin\theta, r\cos\theta$$
$$= \langle 0, 0, r\cos^2\theta + r\sin^2\theta\rangle$$
$$= \langle 0,0,r\rangle \quad \text{But oriented down.}$$

$\bar{v} = \langle z, y^2, x^2\rangle$
$= \langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle$

$\bar{v}\cdot(\bar{r}_r \times \bar{r}_\theta) =$
$\langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle \cdot \langle 0,0,r\rangle = -r^3\cos^2\theta$

S_1' So, $-\rho \int_0^{2\pi}\int_0^2 r^3\cos^2\theta\,dr\,d\theta = -\rho\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \int_0^2 r^3\,dr$ etc.

S_2' : $\bar{r}_2(r,\theta) = \langle r\cos\theta, r\sin\theta, 1\rangle$
$$\bar{r}_r = \langle \cos\theta, \sin\theta, 0\rangle \ \cos\theta, \sin\theta$$
$$\times\ \bar{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle - r\sin\theta, r\cos\theta$$
$$\langle 0,0,r\cos^2\theta + r\sin^2\theta\rangle = \langle 0,0,r\rangle \ \text{up}$$

$\bar{v}\cdot(\bar{r}_r \times \bar{r}_\theta) = \langle 0, r^2\sin^2\theta, r^2\cos^2\theta\rangle\cdot\langle 0,0,r\rangle$
$= r^3\cos^2\theta$

S_2': So $\rho\int_0^{2\pi}\int_0^2 r^3\cos^2\theta\,dr\,d\theta$

$S_1' + S_2' = 0$! (opposite signs from opposite orientation

S_3' sides : $\bar{r}_3(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$
$$\bar{r}_{3\theta} = \langle -2\sin\theta, 2\cos\theta, 0\rangle \ -2\sin\theta, 2\cos\theta$$
$$\bar{r}_{3z} = \langle 0, 0, 1, 0, 0$$
$$\langle 2\cos\theta, 2\sin\theta, 0\rangle = \bar{r}_{3\theta}\times\bar{r}_{3z}$$

$\bar{v}(\theta, z)$
$= \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle$ $\bar{v} = \langle z, y^2, x^2\rangle$

$\bar{v}\cdot(\bar{r}_{3\theta}\times\bar{r}_{3z}) = \langle z, 4\sin^2\theta, 4\cos^2\theta\rangle\cdot\langle 2\cos\theta, 2\sin\theta, 0\rangle$

$= 2z\cos\theta + 8\sin^2\theta\cos\theta + 0$ $dz\,d\theta$

So, $\int_0^1\int_0^{2\pi}(2z\cos\theta + 8\sin^2\theta\cos\theta)\,d\theta\,dz = 0$

So Flux is zero!

$$= \int_0^1 \left[2z\sin\theta + 8\frac{\sin^3\theta}{3}\right]_0^{2\pi} dz = \int_0^1 0\,dz = 0$$

$$\bar{F}\cdot d\bar{S} = \bar{F}(\bar{r}(u,v))\cdot(\bar{r}_u\times\bar{r}_v)\,du\,dv$$

$$= \bar{F}\cdot\bar{n}\,dS = \bar{F}\cdot\frac{\bar{r}_u\times\bar{r}_v}{\|\bar{r}_u\times\bar{r}_v\|}\cdot\|\bar{r}_u\times\bar{r}_v\|\,du\,dv$$

$\bar{n}$ = unit normal

Jacobian for change of variables.

$$\bar{F}\,dS$$

$$= \bar{F}\,\|\bar{r}_u\times\bar{r}_v\|\,du\,dv$$

Apr 30-10:52 AM