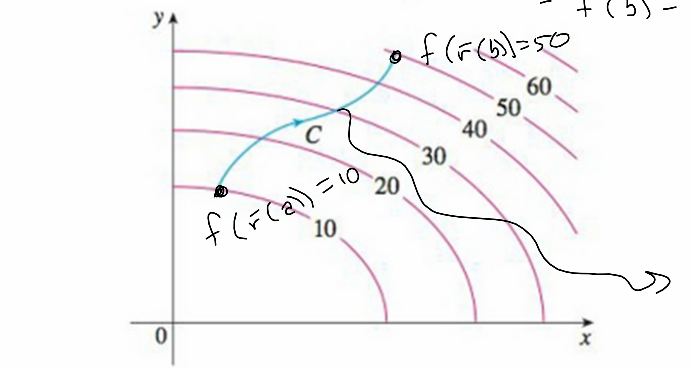
1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.

$= f(b) - f(a) = f(\vec{r}(b)) - f(\vec{r}(a))$

$= 50 - 10 = 40$



$\vec{r}(t)$

16.3 Fundamental Theorem of Line Integrals.

12–18 (a) Find a function f such that $\boxed{\mathbf{F} = \nabla f}$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

13. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t \rangle$, $0 \leq t \leq 1$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{r}'(t)\,dt = \int_C \nabla f \cdot \bar{r}'(t)\,dt = f(\bar{r}(1)) - f(\bar{r}(0))$$

$$f_x = xy^2 \Rightarrow f = \tfrac{1}{2}x^2y^2 + g(y)$$
$$f_y = x^2y \Rightarrow f = \tfrac{1}{2}x^2y^2 + g(x)$$
$$\Rightarrow g(x) = g(y) = \text{constant} \overset{\text{SET}}{=} 0$$

$$f(x,y) = \tfrac{1}{2}x^2y^2 = \tfrac{1}{2}\left(t + \sin\left(\tfrac{\pi}{2}t\right)\right)^2\left(t + \cos\left(\tfrac{\pi}{2}t\right)\right)^2$$

$$\int \nabla f \cdot d\bar{r} = \int_0^1 \nabla f \cdot d\bar{r} = f(\bar{r}(1)) - f(\bar{r}(0))$$
$$= \tfrac{1}{2}\left[\left(1 + \sin\tfrac{\pi}{2}\right)^2\left(1 + \cos\tfrac{\pi}{2}\right)^2 - (0 + \sin 0)^2(\sim\sim)^2\right]$$
$$= \tfrac{1}{2}[1+1]^2[1+0]^2 = \tfrac{1}{2}\cdot 4 = 2.$$

16-4 Green's Theorem.

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$\bar{F} = \langle P, Q \rangle$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} \text{curl } \bar{F}\, dA,$$ where

$\bar{F} = \langle P, Q, 0 \rangle$

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2\,dx + 2x^2y\,dy$,
C is the triangle with vertices (0, 0), (2, 2), and (2, 4)

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11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

12. $\mathbf{F}(x, y) = \langle y^2 \cos x,\, x^2 + 2y \sin x \rangle$, $= \langle P, Q \rangle \Rightarrow$ $Q_x = 2x + 2y\cos x$, $P_y = 2y\cos x$ $\Rightarrow Q_x - P_y = 2x$

C is the triangle from (0, 0) to (2, 6) to (2, 0) to (0, 0)

(0,0) to (2,6)

$(1-t)\langle 0,0\rangle + t\langle 2,6\rangle = \langle 2t, 6t\rangle = \bar{r}_1 \Rightarrow \bar{r}_1' = \langle 2,6\rangle$

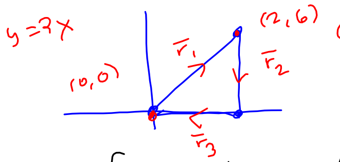
(2,6) to (2,0)

$(1-t)\langle 2,6\rangle + t\langle 2,0\rangle = \langle 2-2t+2t,\, 6-6t+0\rangle = \bar{r}_2 \rightarrow$

$= \langle 2, 6-6t\rangle$ $\qquad \bar{r}_2' = \langle 0,-6\rangle$

(2,0) to (0,0)

$(1-t)\langle 2,0\rangle + t\langle 0,0\rangle = \langle 2-2t, 0\rangle = \bar{r}_3 \Rightarrow \bar{r}_3' = \langle -2, 0\rangle$



Our orientation is clockwise!

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^2\int_0^{3x} (Q_x - P_y)\,dy\,dx$$

$$= \int_0^1 \bar{F}\cdot\langle 2,6\rangle\,dt + \int_0^1 \bar{F}\cdot\langle 0,-6\rangle\,dt + \int_0^1 \bar{F}\cdot\langle 2,0\rangle\,dt$$

w/o green's

Still need to express $\bar{F}(x,y)$ as $\bar{F}(\bar{r}(t))$ for all 3 line segments

$\bar{F}(x,y) = \langle y^2\cos x,\, x^2+2y\sin x\rangle$

$$\int_C \bar{F}\cdot d\bar{r} = \sum_{k=1}^{3}\int_{C_k}\bar{F}\cdot d\bar{r} = \int_0^1 \langle (6t)^2\cos(2t),\, (2t)^2 + 2(6t)\sin(2t)\rangle \cdot \langle 2,6\rangle\,dt$$

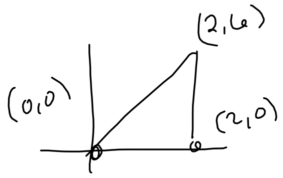
$= \langle 2t, 6t\rangle = \bar{r}_1$ $\qquad + \int_0^1$ etc.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$Q_x = 2x + 2y\cos x$

$P_y = 2y\cos x$

$m = \frac{6-0}{2-0} = \frac{6}{2} = 3 \Rightarrow y = 3(x-0)+0 = 3x$



$$= \int_0^1\int_0^{3x} (2x + 2y\cos x - 2y\cos x)\,dy\,dx$$

$$= \int_0^1\int_0^{3x} 2x\,dy\,dx = \int_0^1 x^2\Big]_0^{3x}\,dx$$

$$= \int_0^1 (3x)^2\,dx = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = \boxed{3}$$

$d\bar{S}$ -vs- dS

Bold-faced S in the book

italics S in book

$$d\bar{S} = \bar{n}\,dS = \bar{n}\,\|\bar{r}_u \times \bar{r}_v\|\,dA$$

$$= \frac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|}\,\|\bar{r}_u \times \bar{r}_v\|\,dA$$

$x+y=3,\ x+y=-2,$
$x+2y-z=6, x+2y-z=1,$
$2x-2y-z=-3,$ and $2x-2y-z=5$

$|\bar{u}\cdot(\bar{v}\times\bar{w})|$

$u = x+y$, $v = x+2y-z$, $w = 2x-2y-z$

Want $\bar{r}(u,v,w) = \langle x,y,z\rangle$

want $x = x(u,v,w)$, $y = y(u,v,w)$

$$\begin{aligned} x+y \quad &= u \\ x+2y-z &= v \\ 2x-2y-z &= w \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & u \\ 1 & 2 & -1 & v \\ 2 & -2 & -1 & w \end{array}\right] \overset{\dots}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$$

Don't have to invert this. The Jacobian for the change of variables from x,y,z to u,v,w is the determinant (absolute value) of the inverse matrix of the above matrix. But the determinant of the inverse is the reciprocal of the original!

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & u \\ 1 & 2 & -1 & v \\ 2 & -2 & -1 & w \end{array}\right]$$

$$\left| 1(-4) - 1(-1-(-2)) \right| = |-4-(1)| = |-5| = 5$$

$$\Longrightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{5}\ !$$

For these LINEAR transformations, you never need the $x = x(u,v,w), \dots$ bit,

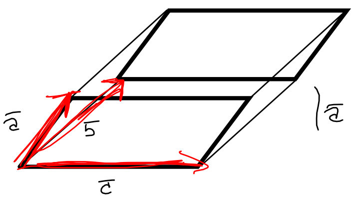
Now Find The Volume!

$$\int_{-2}^{3}\int_{1}^{6}\int_{-3}^{5} 1\cdot\frac{1}{5}\,dw\,dv\,du$$

$$= \frac{1}{5}\int_{-2}^{3} du \int_{1}^{6} dv \int_{-3}^{5} dw =$$

$$\frac{1}{5}[3-(-2)][6-1][5-(-3)]$$

$$= \cancel{\frac{1}{5}}[\cancel{5}][5][8] = 40$$



$|\bar{a}\cdot(\bar{b}\times\bar{c})|$