

**INVERSE THEOREM: THE Pinnacle of Calculus III**

Recall Stokes' Theorem:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$$

It's a generalization of Green's Theorem, and both Green's and Stokes' call on the Fundamental Theorem for Line Integrals, which is more called on FTC II. It all goes back to FTC II. Continuity of the integral guarantees differentiability of the integral, and the integral is the reverse of the differentiation operation.

FTC II can be defined integral depends only on evaluation of the antiderivative on the boundary with a tiny abstraction, which gives us an excellent synonym "Net Change Theorem". It's just really cool that Stokes comes along and generalizes everything. The boundary of a surface is a space curve, so much the same way that the boundary of an interval in FTC II consists of its endpoints.

**STOKES' THEOREM:** Let  $S$  be an oriented piecewise smooth surface that is bounded by a piecewise smooth, positively oriented curve  $C$ . Let  $\mathbf{F}$  be a vector field whose components have continuous partial derivatives on an open region that contains  $S$  and  $C$ . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$$

The line integral is taking  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$  and  $\mathbf{r}'(t)$  is tangent to  $C$  at  $t$ . We measure how to have "parallel" or the boundary  $\mathbf{F}$  is and whether it opposes the orientation of the curve  $C$  is in the same direction.

In 16.5, we also talked about the component of  $\mathbf{F}$  that's normal to the surface:

$$\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}| \cos \theta = |\mathbf{F}| \frac{\mathbf{F} \cdot \mathbf{n}}{|\mathbf{F}| |\mathbf{n}|}$$

The line integral is taking  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$  and  $\mathbf{r}'(t)$  is tangent to  $C$  at  $t$ . We measure how to have "perpendicular" to the boundary  $\mathbf{F}$  is and whether it's "up" or "down" of the surface ( $t = \text{out}, - = \text{in}$ ).

This is, in fact, another way of expressing Green's Theorem, this time in terms of

This generalizes to 3 dimensions as follows:

**THE DIVERGENCE THEOREM, IN ALL ITS GLORY:**

$$\iiint_D \text{div } \mathbf{F} \, dV = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS$$

Actually, here's its full glory:

**THE DIVERGENCE THEOREM:** Let  $E$  be a simple solid region and let  $\mathbf{F}$  be the boundary surface of  $E$ , given with positive (outward) orientation. Let  $\mathbf{F}$  be a vector field whose component functions have continuous partial derivatives on an open region that contains  $E$ . Then

$$\iiint_E \text{div } \mathbf{F} \, dV = \iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS$$

We won't be too worried in this, our first look, about confirming the hypothesis. I wouldn't worry too much about the definition of "simple solid region." In higher analysis, you'll spend a lot of time talking about "convex, simply connected" regions, and the like, and you'll build up more general regions as unions of convex sets, when you try to prove more general results in Advanced Calculus.

We're generally going to be OK, always and everywhere, with the main thing to worry about is the domain of the vector field, which is usually the best indicator of where  $\mathbf{F}$  might NOT have continuous partial derivatives. Look for open sets and domains by now, usually. There can be things you do, if they fall within the region  $E$ , which is the main thing that can make these theorems fail.

$$\mathbf{r}'(t) = \langle \dot{x}, \dot{y}, \dot{z} \rangle = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$$

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**EXAMPLE 1** Find the flux of the vector field  $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$  over the unit sphere  $x^2 + y^2 + z^2 = 1$ .

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \text{div } \mathbf{F} \, dV$$

$\mathbf{F}(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle$   
 $\mathbf{n} = \langle \cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi \rangle$   
 $\mathbf{F} \cdot \mathbf{n} = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle \cdot \langle \cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi \rangle = \sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta - \cos^2 \phi = \sin^2 \phi - \cos^2 \phi = -\cos(2\phi)$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi -\cos(2\phi) \sin \phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[ \frac{1}{2} \sin(2\phi) + \frac{1}{2} \sin^3 \phi \right]_0^\pi \, d\theta = \int_0^{2\pi} \left[ \frac{1}{2} \sin(2\pi) + \frac{1}{2} \sin^3 \pi - \left( \frac{1}{2} \sin(0) + \frac{1}{2} \sin^3 0 \right) \right] \, d\theta = \int_0^{2\pi} 0 \, d\theta = 0$$

Meh. Too slow and clumsy. Too many errors, also.

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**EXAMPLE 1** Find the flux of the vector field  $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$  over the unit sphere  $x^2 + y^2 + z^2 = 1$ .

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV$$

$\mathbf{F}(x, y, z) = \langle z, y, x \rangle$   
 $\text{div } \mathbf{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 0 + 1 + 1 = 2$

$$\iiint_E \text{div } \mathbf{F} \, dV = \iiint_E 2 \, dV = 2 \cdot \text{Volume}(E) = 2 \cdot \frac{4}{3}\pi = \frac{8}{3}\pi$$

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