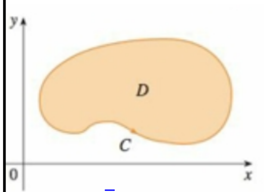
Stokes' Theorem.

We generalize Green's Theorem.



Recall from Section 16.4:

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$\bar{F}\cdot d\bar{r} = \bar{r}'(t)\,dt$ $\qquad \bar{r}(t) = \langle x(t), y(t)\rangle$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cancel{d}A$$

$\bar{F} = \langle P, Q\rangle\ dy\,dy$

$= \langle P(x(t),y(t)),\ Q(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$

I keep trying to express this as a "curl" sort of thing, bringing you guys back to the vector notation and the standard cross product we see, over and over, in the integrand.

$= (P\,x'(t) + Q\,y'(t))\,dt$

$= P\,x'(t)\,dt + Q\,y'(t)\,dt$

$= P\,dx + Q\,dy$

$P = P(x,y) = P(x(t), y(t))$

$Q = Q(x,y) = Q(x(t), y(t))$

$\bar{F} = \langle P, Q, 0\rangle$

$\bar{r} = \bar{r}(t) = \langle x(t), y(t), 0\rangle \implies d\bar{r} = \langle x'(t), y'(t), 0\rangle\,dt$

$\mathrm{curl}(\bar{F}) : \langle 0, 0, Q_x - P_y\rangle$

$\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \rangle, \frac{d}{dx}, \frac{d}{dy}$

$\times \langle P, Q, 0\rangle, P, Q$

$\langle 0, 0, Q_x - P_y\rangle$

$\iint_{\mathcal{D}} \bar{F}\cdot d\bar{S} = \iint \bar{F}\cdot\bar{n}\,dS$

$\dfrac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|}$

$\int_C \bar{F}\cdot d\bar{r} = \iint_{\mathcal{D}} \mathrm{curl}\,\bar{F}\cdot d\bar{S}$

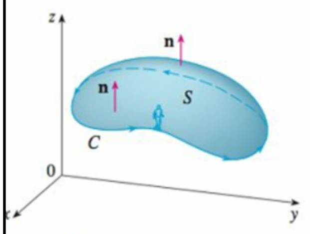


FIGURE 1

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathrm{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

$\mathrm{Curl}\,\bar{F} =$

It's hard to keep my mouth shut (so I don't) about the integral on the left, because I KNOW we wanted to state GREEN'S THEOREM in this language. Now this curl stuff requires a 3-D vector field, **F**. But we can make Green's work in 3-D just by adding a trivial 0 in the 3rd component of **F**.

Stokes' Theorem INCLUDES Green's Theorem as a special case! So it's this huge sledgehammer that covers everything.

$\bar{T} = \dfrac{\bar{r}'}{\|\bar{r}'\|}$ $\qquad ds = \|\bar{r}'\|\,dt$

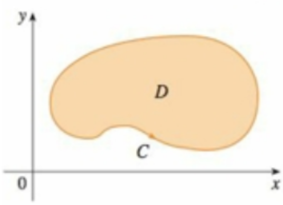
Since I don't know how to say this any better:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds \quad \text{and} \quad \iint_S \mathrm{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_S \mathrm{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

$\dfrac{\bar{r}_u\times\bar{r}_v}{\|\bar{r}_u\times\bar{r}_v\|}$ $\qquad \|\bar{r}_u\times\bar{r}_v\|\,du\,dv$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of **F** is equal to the surface integral of the normal component of the curl of **F**.

$\iint_S \mathrm{curl}\,\bar{F}\cdot(\bar{r}_u\times\bar{r}_v)\,du\,dv$



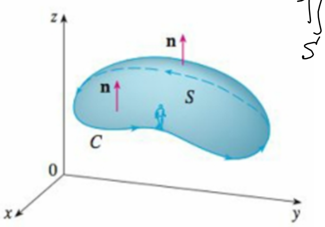
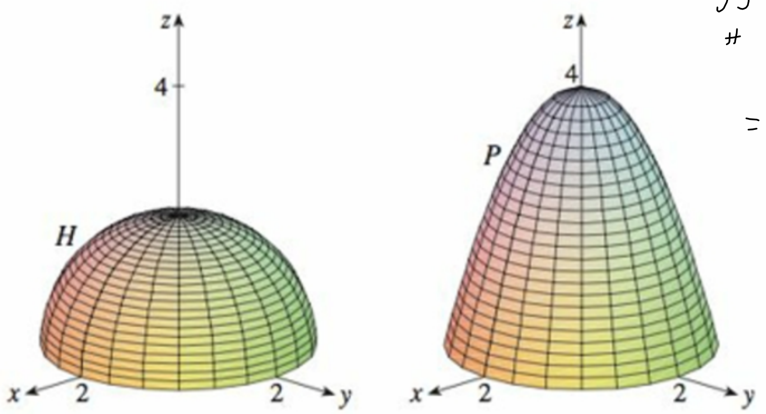


FIGURE 1

Dec 12-7:23 PM

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



$$\iint_H \operatorname{curl}\bar{F} \cdot d\bar{S} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \iint_P \operatorname{curl}\bar{F} \cdot d\bar{S}, \text{ by Stokes' Thm.}$$

2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = 2y \cos z\, \mathbf{i} + e^x \sin z\, \mathbf{j} + xe^y\, \mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upward

2*y(

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T}\, ds = \int_C \bar{F} \cdot \bar{r}'\, dt$$

$$\bar{F} = \langle 2y\cos z, e^x \sin z, xe^y \rangle = \langle 2 \cdot 3\sin t \cos 0, e^x \sin 0, 3\cos t\, e^{3\sin t} \rangle$$

$$= \langle 6\sin t, 0, 3\cos t\, e^{3\sin t} \rangle$$

$$\bar{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \qquad 0 \leq t \leq 2\pi$$

$$\bar{r}'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$$

$$\bar{F} \cdot d\bar{r} = \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \langle 6\sin t, 0, 3\cos t\, e^{3\sin t} \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle\, dt$$

$$= (-18\sin^2 t + 0 + 0)\, dt = -18\left(\frac{1 - \cos(2t)}{2}\right) dt$$

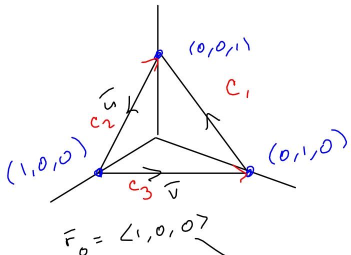
$$= 9(\cos(2t) - 1)\, dt = (9\cos(t) - 9)\, dt$$

and $$\int_0^{2\pi} (9\cos t - 9)\, dt = \Big[9\sin t - 9t\Big]_0^{2\pi} = -18\pi$$

§16.8

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1)



$\bar{F}_0 = \langle 1,0,0\rangle$

$\int_C \bar{F}\cdot d\bar{r}$ Need to build 3 lines & eval 3 line integrals, w/o Stokes.

Eq'n of plane:

$\bar{u} = \langle -1,0,1\rangle$ $-1,0$

$\times\ \bar{v} = \langle -1,1,0\rangle$ $-1,1$

$\langle -1,-1,-1\rangle = \bar{n}$

$\bar{n}\cdot\langle x-1, y, z\rangle = 0$

$-1(x-1) - 1(y) - 1z = 0$

$-x+1-y-z=0$

$z = 1-x-y$ or $x+y+z=1$

↳ characterizes the surface

Now, $\iint_S \bar{F}\cdot d\bar{S} =$

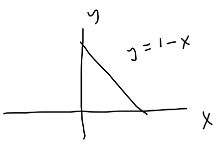
$\iint_S \bar{F}\cdot\bar{n}\,dA =$

$\bar{r}$: $x=x,\ y=y,\ z=1-x-y$

$\bar{r}_x = \langle 1,0,-1\rangle$, $1,0$

$\times\ \bar{r}_y = \langle 0,1,-1\rangle$, $0,1$

$\langle 1,1,1\rangle = \bar{r}_x\times\bar{r}_y$



$$\int_0^1\int_0^{1-x} (\text{curl }\bar{F})\cdot\bar{r}_x\times\bar{r}_y\,dy\,dx$$

scratch: curl $\bar{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\nabla\times\bar{F}$

$\times\ \langle x+y^2,\ y+z^2,\ z+x^2\rangle,\ x+y^2,\ y+z^2$

$\langle 2z, 2x, 2y\rangle = $ curl $\bar{F}$

AHA! HERE'S MY MISTAKE. THE SIGNS ARE ALL WRONG IN THIS CROSS-PRODUCT!

THIS SHOULD BE $\langle -1,-1,-1\rangle$ to match with our orientation

Something's off with the orientation. "outward" normal should be the $\langle 1,1,1\rangle$

$$= \int_0^1\int_0^{1-x} \langle 2z,2x,2y\rangle\cdot\langle 1,1,1\rangle\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \langle 2(1-x-y), 2x, 2y\rangle\cdot\langle 1,1,1\rangle\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}(2-2x-2y+2x+2y)\,dy\,dx = \int_0^1\int_0^{1-x} 2\,dy\,dx = \int_0^1 [2y]_0^{1-x}\,dx$$

$$= 2\int_0^1 (1-x)\,dx = 2\left[x - \tfrac{1}{2}x^2\right]_0^1 = 2\left[1-\tfrac{1}{2}\right] = 1\ !$$

$-$

=

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl }\bar{\mathbf{F}}\cdot d\bar{\mathbf{S}}$$

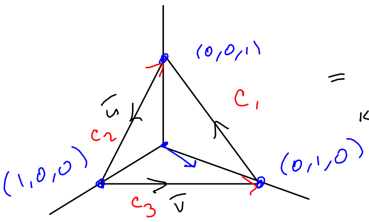
$\bar{r}_x\times\bar{r}_y\,dy\,dx$

$$d\bar{S} = \bar{n}\,dS = \frac{\bar{r}_x\times\bar{r}_y}{\|\bar{r}_x\times\bar{r}_y\|}\,\|\bar{r}_x\times\bar{r}_y\|\,dy\,dx$$

Apr 24-10:40 AM

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$, C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$



The line integral $\int_C \bar{F}\cdot d\bar{r}$

$$= \sum_{k=1}^{3} \int_{C_k} \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

C_1: $(1-t)\langle 0,1,0\rangle + t\langle 0,0,1\rangle$

$= \langle 0, 1-t, 0\rangle + \langle 0,0,t\rangle$

$= \bar{r}(t) = \langle 0, 1-t, t\rangle = \langle x(t), y(t), z(t)\rangle$

& so $\bar{F}(\bar{r}(t)) = \langle 0 + (1-t)^2,\ (1-t) + t^2,\ t + 0^2\rangle$

$= \langle t^2 - 2t + 1,\ t^2 - t + 1,\ t\rangle$

& $\bar{r}'(t) = \langle 0, -1, 1\rangle$

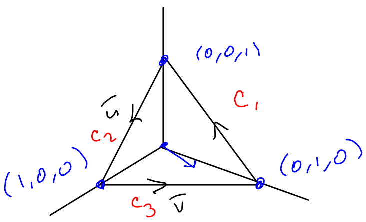
& $\bar{F}\cdot\bar{r}' = \langle t^2-2t+1, t^2-t+1, t\rangle \cdot \langle 0,-1,1\rangle$

$= 0 - t^2 + t - 1 + t = -t^2 + 2t - 1$

$\therefore \int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^1 (-t^2+2t-1)\,dt = \left[-\frac{1}{3}t^3 + t^2 - t\right]_0^1 = -\frac{1}{3} + 1 - 1 = -\frac{1}{3} = \int_{C_1}$

C_2: $\bar{r}(t) = (1-t)\langle 0,0,1\rangle + t\langle 1,0,0\rangle = \langle 0,0,1-t\rangle + \langle t,0,0\rangle$

$= \langle t, 0, 1-t\rangle \implies x(t) = t,\ y(t) = 0,\ z(t) = 1-t$



$\implies \bar{F}(x,y,z) = \langle x+y^2, y+z^2, z+x^2\rangle$

$= \langle t + 0^2, 0 + (1-t)^2, 1 - t + t^2\rangle$

$= \langle t, t^2 - 2t + 1, t^2 - t + 1\rangle$

$\bar{r}'(t) = \langle 1, 0, -1\rangle$

$\implies \bar{F}\cdot\bar{r}' = t + 0 - t^2 + t - 1 = -t^2 + 2t - 1$

& $\int_{C_2} \bar{F}\cdot d\bar{r} = \int_0^1 (-t^2+2t-1)\,dt = -\frac{1}{3}t^3 + t^2 - t\Big]_0^1 = \boxed{-\frac{1}{3}t^3 = \int_{C_2}}$

C_3: $(1-t)\langle 1,0,0\rangle + t\langle 0,1,0\rangle$

$= \langle 1-t, t, 0\rangle = \bar{r}(t) \implies \bar{r}' = \langle -1, 1, 0\rangle$

$\bar{F}(x,y,z) = \langle x+y^2, y+z^2, z+x^2\rangle$

$\implies \bar{F}(\bar{r}(t)) = \langle 1 - t + t^2, t + 0^2, 0 + t^2 - 2t + 1\rangle$

$= \langle t^2 - t + 1, t, t^2 - 2t + 1\rangle \implies$

$\bar{F}\cdot\bar{r}' = \langle t^2 - t + 1, t, t^2 - 2t + 1\rangle \cdot \langle -1, 1, 0\rangle$

$= -t^2 + t - 1 + t = -t^2 + 2t - 1$

$\implies \int_{C_3} \bar{F}\cdot d\bar{r} = \int_0^1 (-t^2 + 2t - 1)\,dt = \dots = -\frac{1}{3} = \int_{C_3}$

So $\int_C \bar{F}\cdot d\bar{r} = \sum_{k=1}^{3} \int_{C_k} \bar{F}\cdot d\bar{r} = -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -1.$

So, I'm off by a sign.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\,\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

Not too interested in your doing this.

$$\int_C f\,\nabla g \cdot d\bar{r} = \iint_{S'} \text{curl}\,(f\,\nabla g) \cdot dS'$$

$$f = (x, y, z), \quad g = (u, v, w)$$

$$f\,\nabla g = \langle f u_x, f v_y, f w_z \rangle$$

$$\text{curl}\,(f\,\nabla g) = \nabla \times f\,\nabla g$$

$$\langle \tfrac{d}{dx}, \tfrac{d}{dy}, \tfrac{d}{dz} \rangle \; \tfrac{d}{dx}, \tfrac{d}{dy}$$

$$= \times \langle f u_x, f v_y, f w_z \rangle \; f u_x, f v_y$$

$$\langle \tfrac{d}{dy}(f w_z) - \tfrac{d}{dz}(f v_y), \tfrac{d}{dz}(f u_x) - \tfrac{d}{dx}(f w_z), \tfrac{d}{dx}(f v_y) - \tfrac{d}{dy}(f u_x) \rangle$$

$$= f_y w_z + f w_{zy} - f_z v_y - f v_{yz}, f_z u_x + f u_{xz} - f_x w_z - f w_{zx}, f_x v_y + f v_{yx} - f_y u_x - f u_{xy} \rangle$$

Dec 18-8:58 PM

Dec 22-7:00 PM