$\bar{F} = \langle P, Q, R \rangle \Rightarrow \text{curl}\,\bar{F} = \nabla \times \bar{F}$

$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

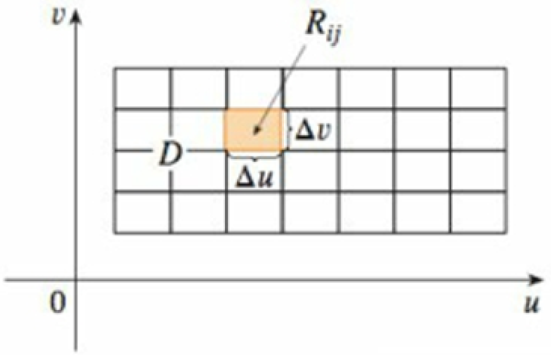
$\times \langle P, Q, R \rangle, P, Q$

$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ might be worth putting on a cheat sheet. So might the 2-D version

$\langle 0, 0, Q_x - P_y \rangle$

$\langle P, Q \rangle$ has no curl, strictly speaking

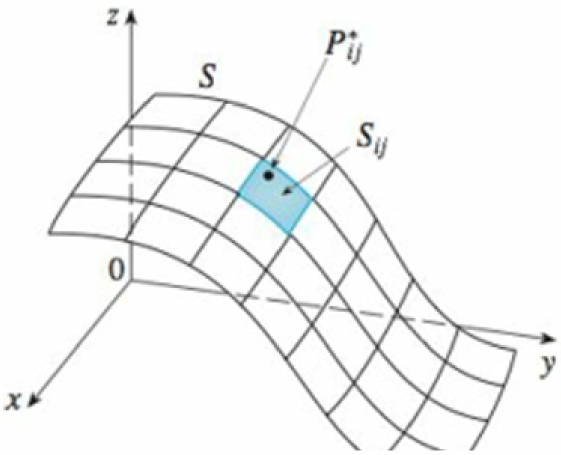
$\langle P, Q, 0 \rangle$



$\downarrow \mathbf{r}$

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

$$(u, v) \in D$$



$$\iint_S f(x, y, z)\, dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v \rightsquigarrow \|\bar{r}_u \times \bar{r}_v\|\, du\, dv$$

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

All this stuff is true, but I'm not sure I'd waste much brain power on memorizing the form it takes.

$\langle x, y, g(x,y)\rangle = \langle x, y, z\rangle$

and so we have $\qquad \mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$

$\bar{r}_x = \langle 1, 0, \frac{\partial g}{\partial x}\rangle, 1, 0$

$\times\ \bar{r}_y = \langle 0, 1, \frac{\partial g}{\partial y}\rangle, 0, 1$

$\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle = \bar{r}_x \times \bar{r}_y$

Thus

3 $$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and $$||\mathbf{r}_x \times \mathbf{r}_y|| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$\Rightarrow \|\bar{r}_x \times \bar{r}_y\| =$

Therefore, in this case, Formula 2 becomes

4 $$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_D f\left(x, y, g(x,y)\right)\left\|\bar{r}_x \times \bar{r}_y\right\| dy\, dx \text{ or}$$

$$= \iint_D f\left(x, y, g(x,y)\right)\left\|\bar{r}_x \times \bar{r}_y\right\| dx\, dy$$

is how I think of it, whenever I have this situation:

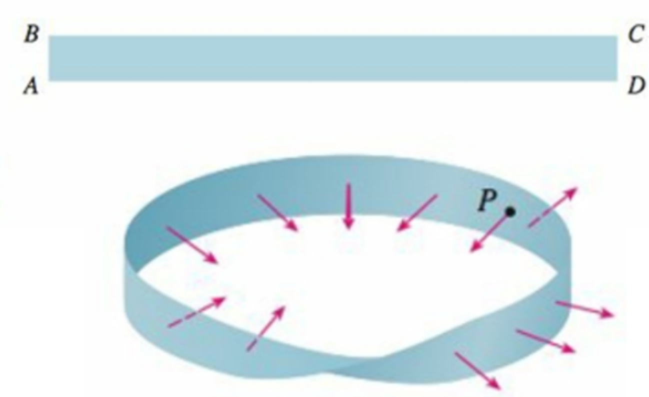
$$x = x \qquad y = y \qquad z = g(x, y)$$

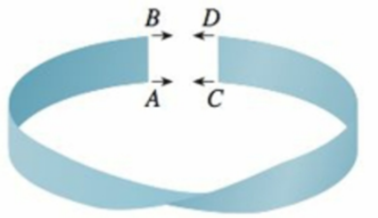
The thing I'm cheating you of in this is the double integral should be an iterated integral if I'm going to use the $dx\ dy$ or $dy\ dx$ instead of the dA. I just want you to *see* what the dA is.

Apr 22-6:38 PM

Oriented/Orientable surfaces. You need to have 2 sides. "top" and "bottom" or "inside" and "outside."

We illustrate with a non-example: the Mobius strip is not orientable, because it only has one "side."





As you go around, you're on the inside and then you're on the outside, without changing sides, so to speak.

There are 2 unit normals associated with any point on an oriented surface

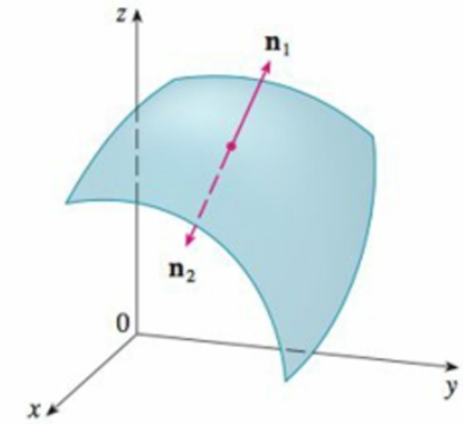


FIGURE 6

$\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z)

By convention, we work with the *outward* normal. The basic idea is when in doubt, check and see what direction your

$$\overline{r}_x \times \overline{r}_y$$

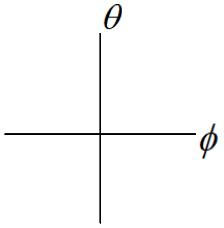
is facing. If it's generally in the same direction as a position vector corresponding to a point on the surface, then you got the outward normal. The inward normal will tend to point towards the origin

The greek letter ϕ stands for the English f, and the greek letter θ stands for the English q. f before q in the alphabet, so ϕ comes before θ when we write

$$\mathbf{r}(\phi, \theta)$$

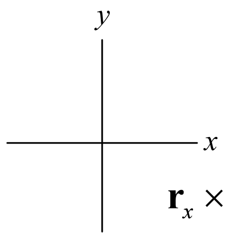
And the comes first when we find **n** with our standard cross-product. If we do the cross product in the reverse order, we get **- n** which is generally the inner product, under these conventions (including right-hand-rule when we're in the parameter domain:

θ



$\mathbf{r}_\phi \times \mathbf{r}_\theta$

In the back of my head, I've always put θ before ϕ, because θ was my first Greek letter, EVER. Recall, u comes before v. Same procedure.



$\mathbf{r}_x \times \mathbf{r}_y$

x & y are the parameters is all you need to know.

For a surface $z = g(x, y)$ $\quad \mathbf{n} = \dfrac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$

$= \dfrac{\bar{r}_x \times \bar{r}_y}{\|\bar{r}_x \times \bar{r}_y\|}$

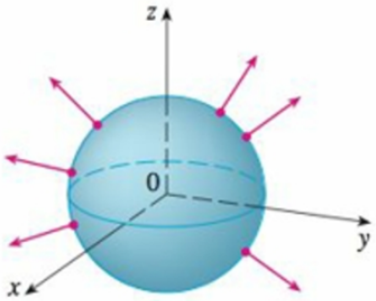
Gee, that's a lot to remember! But no. Just remember:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

as the basic idea, and the big messy formula above, it's just $x = u$ and $y = v$.

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$



the sphere $x^2 + y^2 + z^2 = a^2$.
Positive orientation

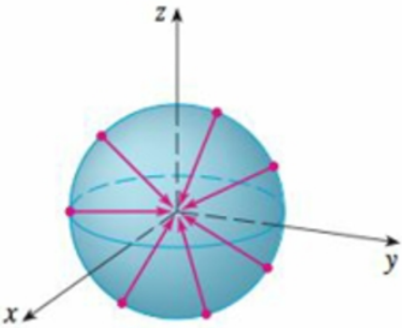


FIGURE 9
Negative orientation

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} =$$

(so the unit normal is EXACTLY the same direction as the position vector, when your surface is a sphere centered at the origin.

Apr 22-7:51 PM

Surface integral (of a vector field over a surface).

We're basically measuring how "in the same direction" the vector field is with the unit normal to the surface. The big application? Heat/electrical/magnetic flux (flow in or out) through the surface, with the vector field being the electrical/magnetic/heat/pollution/oil that we're measuring. All kinds of applications.

That's how you want to think of these dot products. "How much in the same direction are and ?

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$\mathbf{F}$ = vector field

$\mathbf{n}$ = unit (outward) normal to the surface S

dS = increment of area on the surface S.

8 DEFINITION If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of **F** across S.

HERE is where the **S** comes in. It's not the same as the surface *S*. When you see it in **BOLD, No Italics**, it's the increment of area TIMES the outward unit normal, i.e. it's a vector in the direction of the unit normal, whose magnitude is the increment of area.

The increment of area *dS* is a SCALAR.

The punchline, shortly, will be to combine this idea of a surface integral, and, via Green's theorem, relate the surface integral to a *line* integral around the boundary of the surface. Recall, the line integral of a vector field is seeing how much the field is in the same direction as the UNIT TANGENT along the contour!

Weird, wild and wacky stuff. You're getting the keys to the universe here in Calculus III.

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}$$

$ds = |\overline{r}\,'(t)|\,dt$ Line Integral

$dS = \|\overline{r}_\phi \times \overline{r}_\theta\|\,d\phi\,d\theta$ Surface Integral

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

The *dA* is *du dv* but we generally wait to write this, until we sort out the limits of integration in the iterated integral.

It all comes back to that increment of area and change of variables we saw in the previous chapter!

Don't clutter your brain. Just remember

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\theta\,\mathbf{k}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$$

Notice that the magnitude of the cross product need never be computed! It cancels itself out!

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA \\
&= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi\,d\theta \\
&= 2\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta \\
&= 0 + \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta \qquad \left(\text{since } \int_0^{2\pi}\cos\theta\,d\theta = 0\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

In the case of a surface S given by a graph $z = g(x, y)$

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes $\langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle$

10 $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

I don't see the real value in trying to memorize this. I'm more interested in the basic theory, and you don't need to clutter your mind with a special case. Just remember the cross-product idea, and rejoice when the parameters are x and y, because the cross product is pretty straightforward. Why waste time memorizing the final result of the easiest of the calculations?

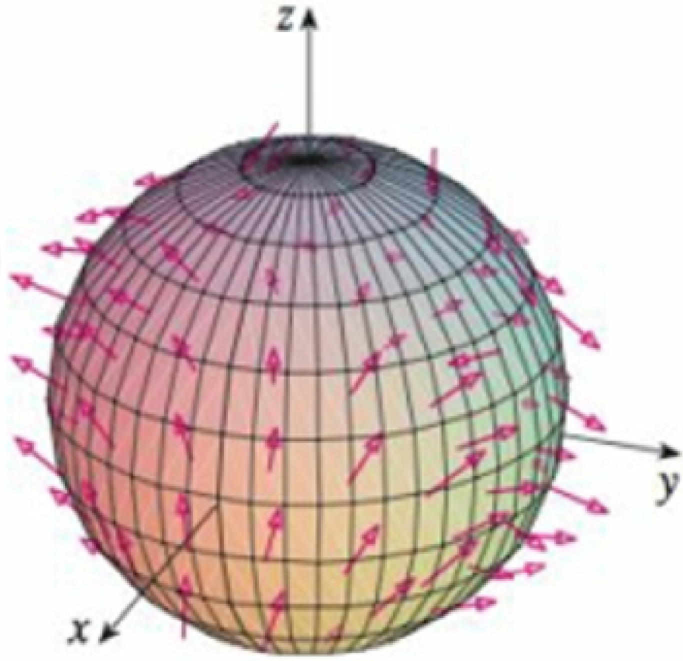


FIGURE 11

Apr 22-8:37 PM