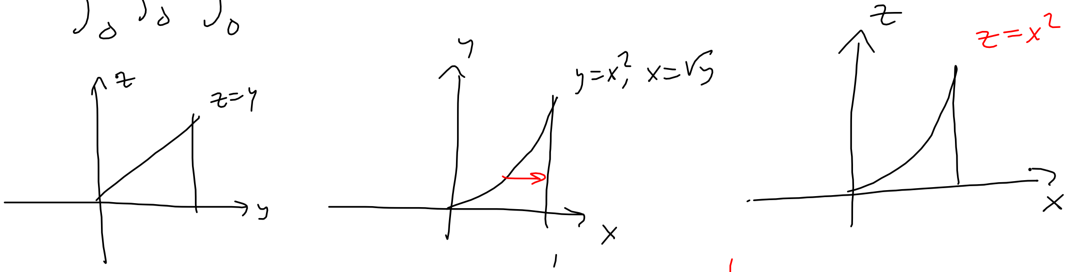
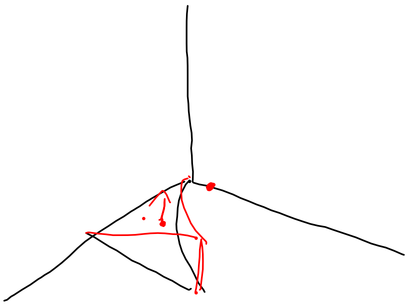
$$\int_0^1\int_0^{x^2}\int_0^y dz\,dy\,dx$$





$$\int_0^1\int_{\sqrt{y}}^1\int_0^y dz\,dx\,dy$$

16.5

CURL AND DIVERGENCE

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \langle P, Q, R\rangle$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$$

This should make you think of torque, and curl does, indeed, say something about the tendency of things to rotate.

$$\nabla \times \bar{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R\rangle :$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$\langle P, Q, R\rangle, P, Q$$

$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

3 THEOREM If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

$\nabla \times \langle f_x, f_y, f_z \rangle$

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere.

4 THEOREM If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

That's why we spend a lot of time looking for holes. But just because curl(**F**) is zero doesn't automatically mean that **F** is conservative. But if it's defined everywhere, with continuous second partials everywhere, then yes.

This is the first example where a conservative field **F** was a function of 3 variables. Part (b) takes it to the next (3-variable) level.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(a) cnt^{s}ly dfbl $\forall\ (x,y,z) \in \mathbb{R}^3$

$\nabla \times \vec{F} = \text{curl}(\vec{F})$

$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle, y^2z^3, 2xyz^3$

$\langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle$

(b) $\vec{F} = \nabla f$ for some f, mean:

$f_x = y^2z^3 \Rightarrow f = xy^2z^3 + g(y,z)$

$f_y = 2xyz^3 \Rightarrow f = xy^2z^3 + h(x,z)$

$f_z = 3xy^2z^2 \Rightarrow f = xy^2z^3 + i(x,y)$

$\Rightarrow g(y,z) = h(x,z) = i(x,y) = C, \ \forall\ C \in \mathbb{R}$

& so $f(x,y,z) = xy^2z^3$ works

Other Method

$f_x = y^2z^3 \Rightarrow f = xy^2z^3 + g(y,z) = xy^2z^3 + g(z)$

$\Rightarrow f_y = 2xyz^3 + \frac{\partial g}{\partial y}(y,z) = 2xyz^3 \Rightarrow$

$\frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = g(z)$

$f_z = 3xy^2z^2 = 3xy^2z^2 + g'(z)$

$\Rightarrow g'(z) = 0 \Rightarrow g = C \stackrel{\text{SET}}{=} 0.$

divergence of F

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

curl **F** is a vector field but div **F** is a scalar field.

11 THEOREM If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

(Apply Clairaut's 3 times.)

$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$div\left(curl\left(\bar{F}\right)\right) = \nabla \bullet \left(\nabla \times \bar{F}\right)$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left(\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle \right)$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \frac{\partial}{\partial x}\left(R_y - Q_z\right) + \frac{\partial}{\partial y}\left(P_z - R_x\right) + \frac{\partial}{\partial z}\left(Q_x - P_y\right)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \longrightarrow \frac{\partial^2 P}{\partial z \partial y}$$

$(f \circ g)(x)$ means

$f(g(x))$ is kind of backwards notation

if you think of other operations

$3 \cdot 2 \cdot 5 = 6 \cdot 5 = 30$ left to right

If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f = \nabla \cdot (\nabla f)$$

The ~~Laplacian~~

Divergence of the Gradient.

Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Harmonic Solutions

2-D:

$f(x) = \sin x \cos y$

$\frac{\partial^2 f}{\partial x^2} = -\sin x \cos y$

$\frac{d^2 f}{dy^2} = -\sin x \cos y$

Ugh. No. Don't waste time.

Take everything we just did and apply it to the plane by assigning the 3rd component of **F** to 0.

VECTOR FORMS OF GREEN'S THEOREM

$$\mathbf{r} = \bar{r} = \langle x(t), y(t) \rangle$$

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \langle P(x,y), Q(x,y) \rangle = \langle P(x(t),y(t)), Q(x(t),y(t)) \rangle = \mathbf{F}(\mathbf{r})$$

F's line integral: (of the tangential component of **F**.)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \oint_C \langle P(x(t),y(t)), Q(x(t),y(t)) \rangle \bullet \langle x'(t), y'(t) \rangle dt$$

By the expedient of treating **F** as a vector field in 3-space, with 3rd entry 0, we can re-state Green's Theorem in vector notation as follows:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} =$$

Less parallel ⇒ more torque.

$$\nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P(x,y), Q(x,y), 0 \rangle$$

$$= \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(Q(x,y)), -\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(P(x,y)) \right), \frac{\partial}{\partial x}(Q(x,y)) - \frac{\partial}{\partial y}(P(x,y)) \right\rangle$$

$$= \left\langle 0 - \frac{\partial Q}{\partial z}, -\left(0 - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle 0, 0, Q_x - P_y \rangle$$

→ Torque!

The line integral of the tangential component of F along *C* is the double integral of the vertical component of curl F over the region *D* enclosed by *C*.

That's a double mouthful. It's easier to express, symbolically:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\,dA$$

Recall that TORQUE is a vector perpendicular to the two vectors whose cross product we're taking. Cross product is a "sine" thing, so more parallel means less twisting. Curl of two vectors in the "board" is a vector pointing straight up (or down) out of the board.

Recall the outward normal vector **n**.

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

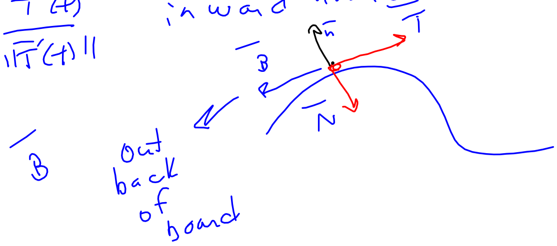
(Obtained by the easiest way to build a vector whose dot product with the unit tangent is 0.)

(It also turns out to be a unit vector in the OPPOSITE direction from the familiar (inward) normal **N**.

$$\bar{T} = \frac{x'(t)}{\|\bar{r}'(t)\|}\,\bar{i} + \frac{y'(t)}{\|\bar{r}'(t)\|}\,\bar{j}$$

$$\bar{N} = \frac{\bar{T}'(t)}{\|\bar{T}'(t)\|}$$

inward normal



$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt = \int_a^b \langle P,Q\rangle \bullet \frac{1}{\|\langle x'(t),y'(t)\rangle\|}\langle -y'(t),x'(t)\rangle\,dt$$

$$= \int_a^b \left[\frac{P(x(t),y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \qquad \bar{F}\cdot d\bar{r}$$

$$= \int_a^b P(x(t),y(t))\,y'(t)\,dt - Q(x(t),y(t))\,x'(t)\,dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \nabla\circ\bar{F}\,dA$$

$$= \iint_D \mathrm{div}(\bar{F})\,dA$$

This last by Green's Thm.

This gives us a second vector form for Green's Thm:

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

Eg in 13
See #35
(& #33)

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \mathrm{div}\,\mathbf{F}(x,y)\,dA$$

outward

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

Monday: Ask me questions. I will be diving into 16.6 lecture.

When we're done lecturing over Chapter 16, it'll be more of a lab setting in class. Clearing up questions, explaining things in more detail. MAYbe even diving into Maxwell's Equations.