16.2 Line Integrals

The Line Integral with respect to Arc Length:

**2 DEFINITION** If $f$ is defined on a smooth curve $C$ given by Equations 1, then the **line integral of $f$ along $C$** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

The increment of arc length is the same as before:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = |\bar{r}\,'(t)|\, dt \qquad \bar{r}(t) = \langle x(t), y(t) \rangle$$

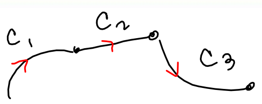
$$= \sqrt{x_t^2 + y_t^2}\, dt$$

Recall the arc-length integral:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b ds$$

If $f$ is (piecewise-) continuous, we can integrate along (the pieces of) $C$ :

**3** $$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$




$$= \sum_{k=1}^{3} \int_{C_k} f(x,y)\, ds$$

Arc length DGAD about direction!

$$\int_C \cdots\, ds = \int_{-C} \cdots\, ds$$

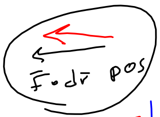
**line integrals of $f$ along $C$ with respect to $x$ and $y$:** $P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C f(x, y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i \qquad \int_C \vec{F} \cdot d\vec{r}$$

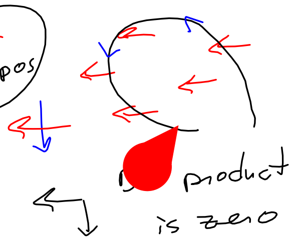
$$\int_C f(x, y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i = \int_C \langle P(x,y), Q(x,y)\rangle \cdot \overbrace{\langle x'(t), y'(t)\rangle\,dt}^{d\vec{r} = \vec{r}'(t)\,dt}$$

$$\int_C f(x, y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$$\int_C f(x, y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$



$\vec{F} \cdot d\vec{r}$ pos



product is zero

$\cos\theta \longleftrightarrow$ Dot Product

The following arise naturally when we evaluate line integrals of vector fields. My main gripe with the notation, here, is they need parentheses. Book way:

$$\int_C P(x, y)\,dx + \int_C Q(x, y)\,dy = \int_C P(x, y)\,dx + Q(x, y)\,dy$$

Dot product is negative.

Mills way: $\int_C \vec{F} \cdot d\vec{r} = \int_C \langle P, Q\rangle \cdot \langle x', y'\rangle\,dt = \int_C \big(P\,x'(t) + Q\,y'(t)\big)\,dt$

$$= \int_C P(x,y)\,\underbrace{x'(t)\,dt}_{dx} + \int_C Q(x,y)\,y'(t)\,dt$$

$x = x(t) \longrightarrow$

$dx = x'(t)\,dt$

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \cancel{\int_C \big(P(x,y)\,dx + Q(x,y)\big)\,dy}$$

ugh! That explain my clumsiness.

$$= \int \big(P(x,y)\,x'(t) + Q(x,y)\,y'(t)\big)\,dt$$

The tough part is the execution: The parametrization of the curve $C$. And when we're integrating with respect to $x$ or $y$ (and *not* w.r.t. arc length), the direction of increasing $t$ gives $C$ an *orientation*, and we speak of "$C$" and " - $C$."

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

Direction (orientation) doesn't matter, when it's w.r.t. arc length.

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

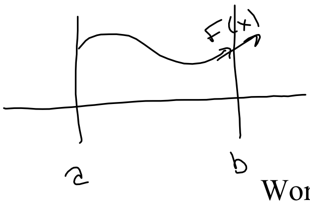
LINE INTEGRALS OF VECTOR FIELDS

Recall the connection between dot product of vectors and cosine of the angle between them. Generally the dot product gives us a measure of "how large a shadow" the one vector casts on the other vector. When vectors are orthogonal (perpendicular), the dot product is ZERO. When they're in the same direction (at least partially), the dot product is positive, and when the angle between them is greater than 90 degrees, the dot product is *negative*.

Work in one dimension. Constant force $F$, distance $D$.

$$W = FD$$

Variable force:




$$\int_a^b F(x)\,dx$$ which simplifies to the above when $F$ is constant.

Work in 2 or 3 dimensions.

The **work** $W$ done by the force field $\mathbf{F}$

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,ds = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

$\mathbf{T}$ gives us a unit vector in the direction of the curve

$$= \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b \vec{F} \cdot d\vec{r}$$

**13 DEFINITION** Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve $C$ given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along $C$** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

$$= \int_a^b \left[P(x(t), y(t), z(t))\,x'(t) + Q(x(t), y(t), z(t))\,y'(t) + R(x(t), y(t), z(t))\,z'(t)\right] dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \langle P, Q, R\rangle$$

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**23–26** Use a calculator or CAS to evaluate the line integral correct to four decimal places. Missing the "*dt*" in this discussion (in Videos).

**23.** $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + \sin y\,\mathbf{j}$ and $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t^2}\,\mathbf{j}$, $1 \leqslant t \leqslant 2$ $\vec{r} = \langle e^t, e^{-t^2} \rangle$

$\Rightarrow \vec{r}\,'(t) = \langle e^t, -2te^{-t^2} \rangle$

$\Rightarrow \vec{F} \cdot d\vec{r} = (xye^t - 2\sin y\, te^{-t^2})\,dt$

$= (e^t e^{-t^2} e^t - 2\sin(e^{-t^2})\, te^{-t^2})\,dt$

$= (e^{2t-t^2} - 2te^{-t^2}\sin(e^{-t^2}))\,dt$

$\int_1^2 (e^{2t-t^2} - 2te^{-t^2}\sin(e^{-t^2}))\,dt$ ______

## 16.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

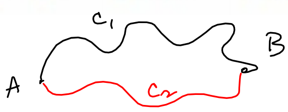
Recall, FTC II: $\int_a^b F'(x)\,dx = F(b) - F(a)$

In the following discussion, we find that FTC II is quite useful when the vector field **F** is a *conservative* field, that is, **F** is the gradient of some function $f$. In that case, ...

$\boxed{2}$ **THEOREM** Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then

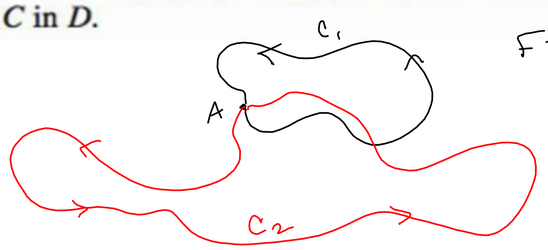

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$




$\int_{C_2} = \int_{C_1}$ is independent of path.

$\boxed{3}$ **THEOREM** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$ if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path $C$ in $D$.




FTC II will give us

$\int_a^a F'(x)\,dx = 0$

$= F(a) - F(a) = 0$

**4 THEOREM** Suppose **F** is a vector field that is continuous on an open connected region $D$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$, then **F** is a conservative vector field on $D$; that is, there exists a function $f$ such that $\nabla f = \mathbf{F}$.

$\vec{F} = \langle f_x, f_y, f_z \rangle = \nabla f$

So, F is conservative if and only if the corresponding line integrals are independent of path.

You want to show it's independent of path? Show it's conservative.

You want to show it's conservative? Prove it's independent of path (not always easy).

Beware of when the vector field is NOT continuous and the domain D contains that discontinuity. Then all bets are off. Example 5 in 16.4 is one of these exceptions. It's because the **F** isn't continuous at the origin.

**5 THEOREM** If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where $P$ and $Q$ have continuous first-order partial derivatives on a domain $D$, then throughout $D$ we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

This is just Clairaut's Theorem.

If F is a gradient for some f, then this equation always holds.

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

$\vec{F}$ conservative $\Rightarrow \vec{F} = \nabla f$

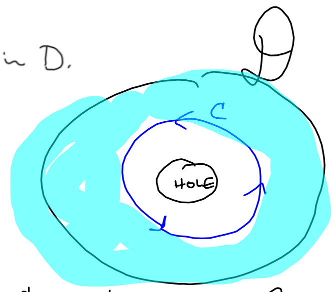
$\vec{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle P, Q \rangle$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$

A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints.

A **simply-connected region** in the plane is a connected region $D$ such that every simple closed curve in $D$ encloses only points that are in $D$. $\Leftrightarrow$ No holes in D.

This is a fancy mathematical way of saying that $D$ has no holes!




C contains a hole (Points not in D)

**6 THEOREM** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region $D$. Suppose that $P$ and $Q$ have continuous first-order derivatives and

$\vec{F} = \langle P, Q \rangle$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

For MOST intents and purposes, Theorem 6 gives us a working converse to Theorem 5. You just want to be careful not to think everything's hunky-dory, when the domain has a hole, or the field is undefined somewhere inside the domain. All this theory requires the field to be nice and continuous (which implies bounded).

smooth

(for most functions we can write down.)

**3–10** Determine whether or not **F** is a conservative vector field. If it is, find a function $f$ such that $\mathbf{F} = \nabla f$.

**7.** $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$ $= \langle f_x, f_y \rangle = \nabla f$ for some $f$?

$= \langle P, Q \rangle$

This technique is basically "variation of parameters" methods used in differential equations, later on.

$P_y = e^x + \cos y$

$Q_x = e^x + \cos y = P_y$

Yes!

Now to find $f$

Technicalities:

Does $\vec{F}$ blow up anywhere?

Nah. It's dif'ble everywhere!

differentiable

$\vec{F} = \langle ye^x + \sin y, e^x + x\cos y \rangle = \langle f_x, f_y \rangle$

Then $f_x = ye^x + \sin y \implies$

Josh

$\int f_x\,dx = \int (ye^x + \sin y)\,dx = ye^x + x\sin y + C$

$= \boxed{ye^x + x\sin y + g(y) = f(x,y)}$

$\implies f_x = ye^x + \sin y + 0$

$f_y = e^x + x\cos y$

$= \frac{d}{dy}\left(ye^x + x\sin y + g(y)\right)$

$= \boxed{e^x + x\cos y + g'(y) = e^x + x\cos y}$

$\implies g'(y) = 0$!

$\implies g(y) =$ KONSTANT $K$.

For our purposes, choose $K \equiv 0$

So $f(x,y) = ye^x + x\sin y$

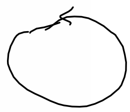
$f_x = ye^x + \sin y$

$f_y = e^x + x\cos y$ ✓

16.4 GREEN'S THEOREM We relate line integrals

**GREEN'S THEOREM** Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



Note: If $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r} = 0$ Think $\int_a^b F' = F(b) - F(a)$

**EXAMPLE 4** Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where $C$ is the boundary of the semiannular region $D$ in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

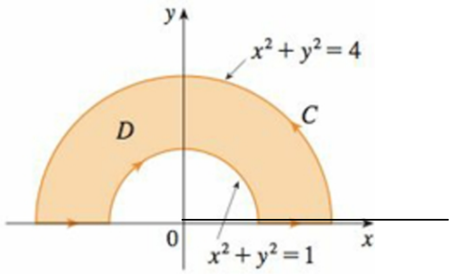



**FIGURE 7**

16.1 See the Announcements

16.2 Tomorrow

16.3 Good whack out of it plus questions.

16.4 Monday.

Answer questions on all of it, Thursday.

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA = \int_0^\pi \int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = \left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$

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