1 DEFINITION Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

2 DEFINITION Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

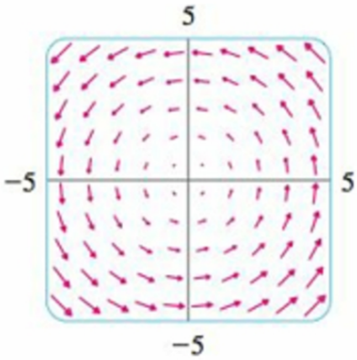


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

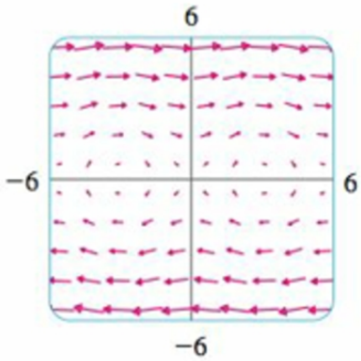


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

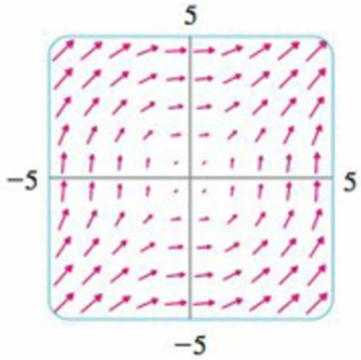


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

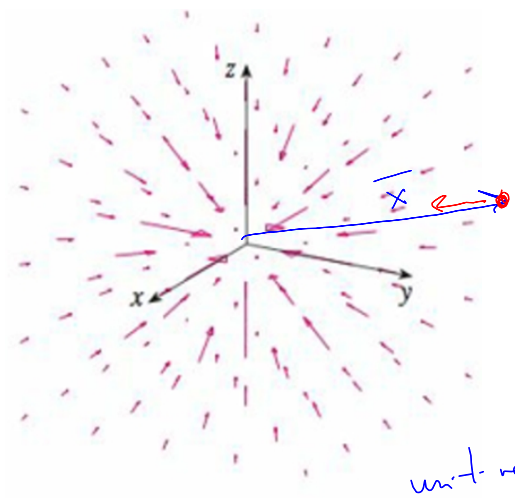


FIGURE 14
Gravitational force field

unit vector in Direction of force

$$= \left(-\frac{mMG}{\|\bar{x}\|^2}\right)\frac{\bar{x}}{\|\bar{x}\|}$$

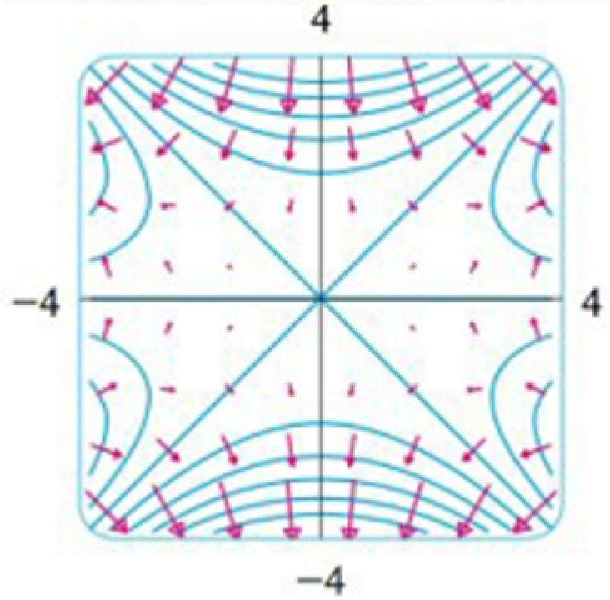
$$|\mathbf{F}| = \frac{mMG}{r^2} \quad \|\bar{x}\|^2$$

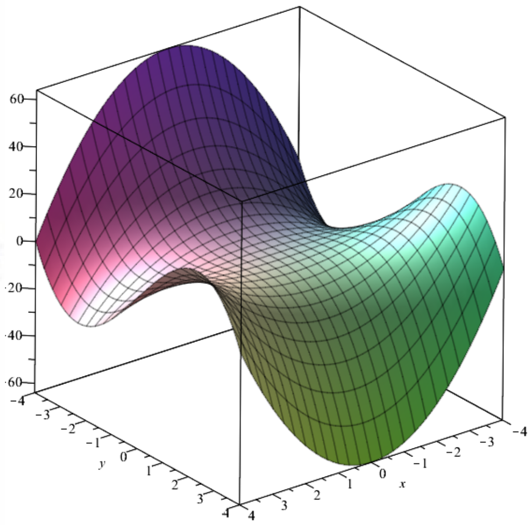
$$-\frac{\mathbf{x}}{|\mathbf{x}|} = -\frac{\bar{r}}{\|\bar{r}\|}$$

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

Force is jointly propor-
tional to the 2 masses &
inversely proportional
to the square of the
distance between them.

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$.





A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field **F** in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

$\langle f_x, f_y, f_z \rangle$

$$\frac{d}{dx}\left[\frac{\text{stuff}}{\sqrt{x^2+y^2+z^2}}\right] = \text{stuff}\left[-\frac{1}{\cancel{2}}\left[(x^2+y^2+z^2)^{-\frac{3}{2}}(\cancel{2}x)\right]\right]$$

$=$

Recall: arc length increment:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad , \text{where } x = x(t),\ y = y(t)$$

$y = f(x)$

$$ds = \sqrt{1 + f'(x)^2}\, dx$$

If the curve C is written this way:

$\bar{r}(t) = \langle x(t), y(t) \rangle$, then

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = |\bar{r}'(t)|\, dt$$

Line segments: ... where $0 \le t \le 1$

15. $\int_C (x + yz)\, dx + 2x\, dy + xyz\, dz$, C consists of line segments from $\boxed{(1, 0, 1) \text{ to } (2, 3, 1)}$ and from $(2, 3, 1)$ to $(2, 5, 2)$

C_1

Thx Josh

$\bar{r}(t) = (1-t)\langle 1, 0, 1\rangle + t\langle 2, 3, 1\rangle = \langle 1-t+2t, 3t, 1-t+t\rangle$

$= \langle t+1, 3t, 1\rangle$

$x = t+1$ $\quad$ $y = 3t$ $\quad$ $z = 1$

$dx = dt$ $\quad$ $dy = 3dt$ $\quad$ $dz = 0$

$$\int_C = \int_{C_1} + \int_{C_2}$$

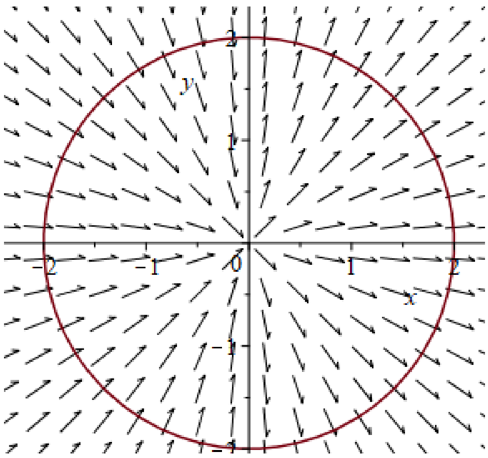
$$\int_{C_1}: \int_0^1 ((t+1) + (3t)(1))\, dt + \int_0^1 2(t+1)\, 3dt + 0$$

etc.

But that's how those go

$\bar{F} = \langle x^2, xy \rangle$

work done: $\int_C \bar{F} \cdot d\bar{r}$ $\quad x = 2\cos t$, $y = 2\sin t$



$d\bar{r} = \bar{r}\,'(t)\,dt$

$\bar{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$\bar{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$\int_0^{2\pi} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \langle 4\cos^2 t, 4\cos t \sin t \rangle \cdot r'(t)$

$= \int_0^{2\pi} \big((4\cos^2 t)(-2\sin t) +$
$(2\cos t)(2\sin t)(2\cos t)\big)dt$