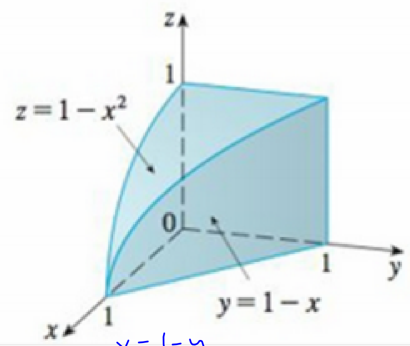
Section 15.6: Write the triple integral in 5 ways.

Want to hit this one, today, and talk about 15.9, today.

34. The figure shows the region of integration for the integral

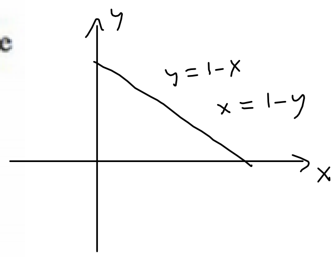
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

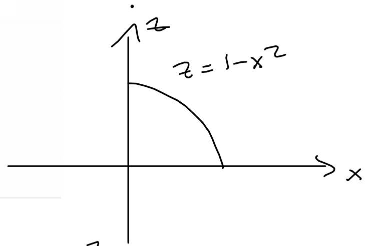
Rewrite this integral as an equivalent iterated integral in the five other orders.

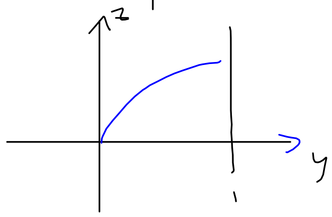


$x = 1 - y$

Some right. Some wrong.







$-(y-1)^2 + 1 = z$

$(y-1)^2 = 1 - z$

$y - 1 = \sqrt{1-z}$

$y = \sqrt{1-z} + 1$

$$\boxed{z = 1 - x^2 = 1 - (1-y)^2 \; ?}$$

dydx or dxdy

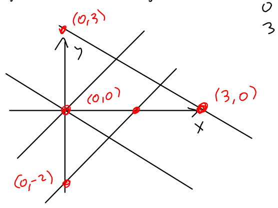
20. $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

$v = x-y=2$ $\quad$ $x-y=0=v$ $\quad$ $x+y=0=u$ $\quad$ $x+y=3=u$

$y=x$ $\quad$ $y=-x$

x	y
0	-2
2	0

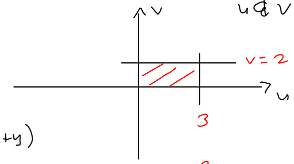
x	0
0	3
3	0



I think the obvious transformation is

$u = x+y$ $\quad$ Want x & y in terms of u & v!

$v = x-y$



$\vec{r}(u,v) = ?$

We have only the inverse operation from uv-plane to xy-plane.

$(x+y)e^{(x-y)(x+y)}$ $\qquad$ $\int_0^2\int_0^3 r(u,v)\,dA$

We need $\vec{r}(u,v) = \langle x,y\rangle = \langle x(u,v), y(u,v)\rangle$ $= \int_0^2\int_0^3 u e^{uv}\,\|\vec{r}_u \times \vec{r}_v\|\,du\,dv$

We have:

$u = x+y$

$v = x-y$

Solve this for x & y!

Substitution:

$x = u-y$ $= u - \left(\tfrac{1}{2}u - \tfrac{1}{2}v\right) = u - \tfrac{1}{2}u + \tfrac{1}{2}v$

$\boxed{= \tfrac{1}{2}u + \tfrac{1}{2}v = x}$

$\Rightarrow v = u-y-y$

$= u-2y$

$v-u = -2y$

$\boxed{\tfrac{1}{2}u - \tfrac{1}{2}v} = \tfrac{1}{2}(u-v) = y$

$\vec{r}(u,v) = \langle \tfrac{1}{2}u + \tfrac{1}{2}v,\ \tfrac{1}{2}u - \tfrac{1}{2}v\rangle = \langle \tfrac{1}{2}u+\tfrac{1}{2}v, \tfrac{1}{2}u - \tfrac{1}{2}v, 0\rangle$

$\|\vec{r}_u \times \vec{r}_v\|$: $\quad \vec{r}_u = \langle \tfrac{1}{2}, \tfrac{1}{2}\rangle$ $\quad$ Can do as a

$\vec{r}_v = \langle \tfrac{1}{2}, -\tfrac{1}{2}\rangle$ $\quad$ cross product OR

simply calculate the determinant of

$$\left\| \begin{matrix} \vec{r}_u \\ \vec{r}_v \end{matrix} \right\| = \left\| \begin{matrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & -\tfrac{1}{2} \end{matrix} \right\| = \left| -\tfrac{1}{4} - \tfrac{1}{4} \right| = \tfrac{1}{2} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$\frac{1}{2}\int_0^2\int_0^3 u e^{uv}\,du\,dv$$

The Jacobian!

MATRIX METHOD FOR getting $x = x(u,v)$, $y = y(u,v)$:

$x+y = u$

$x-y = v$

$$\left[\begin{array}{cc|c} 1 & 1 & u \\ 1 & -1 & v \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & u \\ 0 & -2 & v-u \end{array}\right]$$

Back-substitute OR

take it all the way:

$$\left[\begin{array}{cc|c} 1 & 1 & u \\ 0 & 1 & \frac{u-v}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & u - \frac{u-v}{2} \\ 0 & 1 & \frac{u-v}{2} \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 0 & \frac{u+v}{2} \\ 0 & 1 & \frac{u-v}{2} \end{array}\right]$$

Reduced Row-Echelon

$x = \frac{u+v}{2}$

$y = \frac{u-v}{2}$

$x+y=2$

$x-y=3$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 1 \end{array}\right]$$

Back-sub:

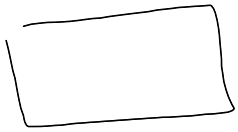
$-2y = 1$ $\quad$ $x+y=2$

$\boxed{y = -\tfrac{1}{2}}$ $\quad$ $x - \tfrac{1}{2} = 2$

$\boxed{x = \tfrac{5}{2}}$

Check:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} \\ \frac{6}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Area dA in terms of u & v is then given by the approximation

$$\|(\Delta u \bar{r}_u) \times (\Delta v \bar{r}_v)\| = \|\bar{r}_u \times \bar{r}_v\| \Delta u \Delta v$$

In 3-D, it's a parallelepiped with $\Delta v \bar{r}_v$, $\Delta u \bar{r}_u$, & $\Delta w \bar{r}_w$ defining it.

Volume is the scalar triple product

$$\left|(\Delta u \bar{r}_u) \cdot \left((\Delta v \bar{r}_v) \times (\Delta w \bar{r}_w)\right)\right|$$

$$= |\bar{r}_u \cdot (\bar{r}_v \times \bar{r}_w)| \Delta u \Delta v \Delta w$$

How do we quickly compute these?

By taking the determinant of this

$$\begin{vmatrix} \bar{r}_u \\ \bar{r}_v \\ \bar{r}_w \end{vmatrix} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$$

Book hides this:

$$\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$ which is the transpose, but still equal.