Section 14.3 Partial Derivatives

§14.3 #s 4, 10, 11, 13, 15, 18, 21, 26, 29, 30, 47, 50, 52, 53, 56, 59, 71*, 78, 81, 83

* Clairaut.

$f_x(a,b) = f_x(a, b) = g'(a)$ where $g(x) = f(x, b)$

$\frac{df}{dx} = ?$

$$\boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}}$$

$$\boxed{\lim_{(h,k) \to (0,0)} f(a+h, b+k)}$$

would like something like this.

$f_y(a,b) = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$

$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

We can no longer just set $\frac{dy}{dx} = 0$, etc.

we <u>Do</u> Have $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ at the same time

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x = f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \qquad D_x f$$

$$f_y = f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

∂ = "del"

$z = f(x, y)$

$\frac{\partial f}{\partial x} = f_x$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$

$= \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

1st x, then y.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

$$f(x,y) = x^2 \sin(xy) + x^4 y^5 \qquad \text{Find } f_x(x,y), f_y(x,y)$$

$f_x = 2x \sin(xy) + x^2 \cos(xy) \cdot y + 4x^3 y^5$

$f_{xy} = 2x \cos(xy) \cdot x - x^2 \sin(xy) \cdot x \cdot y + x^2 \cos(xy) + 20x^3 y^4$

$= 2x^2 \cos(xy) - x^3 y \sin(xy) + x^2 \cos(xy) + 20x^3 y^4$

$= 3x^2 \cos(xy) - x^3 y \sin(xy) + 20x^3 y^4$

$f_y = x^2 \cos(xy) \cdot x + 5x^4 y^4 = x^3 \cos(xy) + 5x^4 y^4$

$f_{yx} = 3x^2 \cos(xy) - x^3 \sin(xy) \cdot y + 20x^3 y^4$

$= 3x^2 \cos(xy) - x^3 y \sin(xy) + 20x^3 y^4 = f_{xy}$

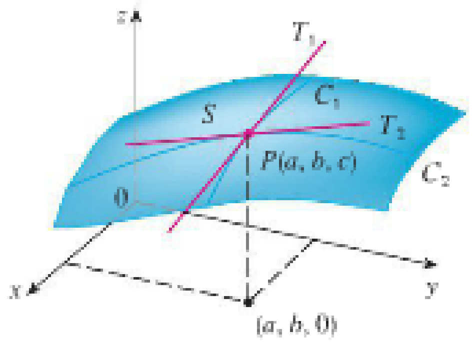


FIGURE 1
The partial derivatives of f at $(a, b$ ... the slopes of the tangents to C_1 and

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

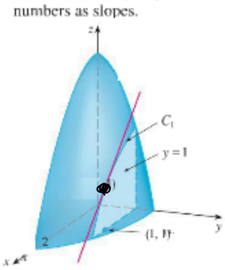
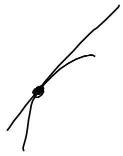


FIGURE 2



f_x vertical plane $\parallel$ to xz-plane

f_y vertical plane $\parallel$ to yz-plane.

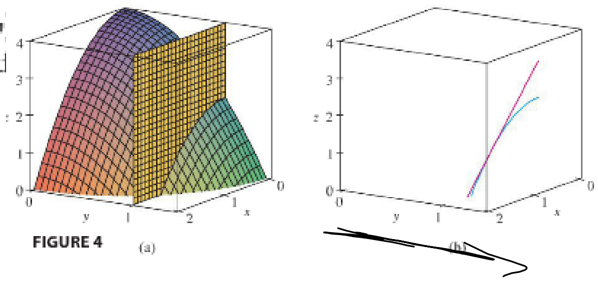


FIGURE 4

Functions of 3 or more variables...

$f(x_1, x_2, \ldots, x_n)$

f_{x_3}, etc. Same deal.

Higher Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

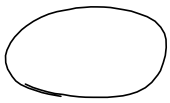
f_{xy} means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

(order is reversed from f_{xy} notation

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f(x,y) = x^2 \sin(xy) + x^4 y^5$$



Find $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

$f_x = 2x\sin(xy) + x^2(\cos(xy))\cdot y + 4x^3y^5 = \underline{2x\sin(xy)} + \boxed{x^2 y\cos(xy)} + 4x^3y^5$

$f_{xx} = 2\sin(xy) + 2x(-\cos(xy))\cdot y + 12xy^5 = 2\sin(xy) + 2x(-\cos(xy))y + 2xy\cos(xy) + x^2y(-\sin(xy))\cdot y$

$f_{xy} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = 2x(\cos(xy))\cdot x + 20x^3y^4$

$f_y = x^2(\cos(xy))\cdot x + 5x^4y^4 = x^3\cos(xy) + 5x^4y^4$

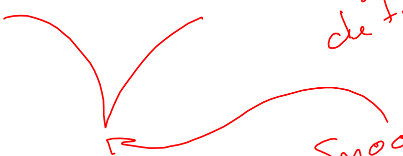
$f_{yy} =$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Smoothness is why.

Pretty much anywhere they're defined.



Not smooth

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **harmonic functions;**

heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

9–10 Draw the graph of f and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.)

Then zoom in until the surface and the tangent plane become indistinguishable.

10. $f(x, y) = e^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy}), \ (1, 1, 3e^{-0.1})$

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

$f_v(40, 15)$

$f(40, 15) = 25$

$f(30, 15) = 16$

$f(50, 15) = 36$

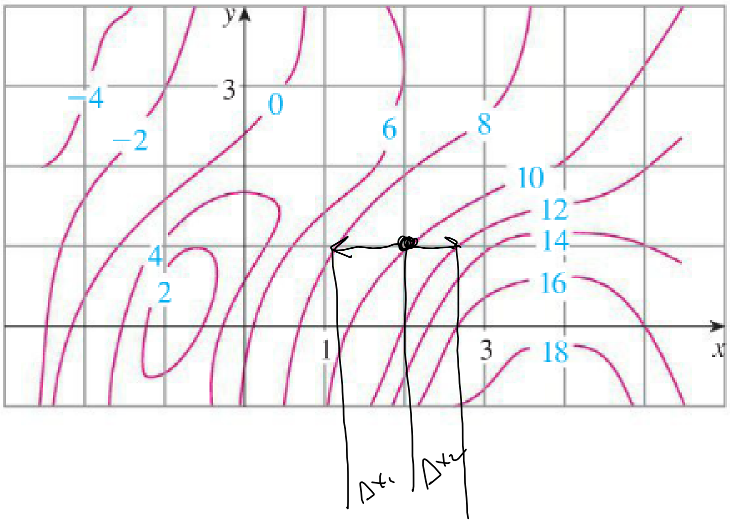
$$\frac{f(40,15) - f(30,15)}{40-30} = \frac{25-16}{10} = \frac{9}{10}$$

$$\frac{f(50,15) - f(40,15)}{50-40} = \frac{36-25}{10} = \frac{11}{10}$$

$$\Rightarrow \frac{\frac{9+11}{10}}{2} = \frac{20}{20} = 1$$

$f_v(40, 15) \approx 1$

10. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



11. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

13–14 Find f_x and f_y and graph f, f_x, and f_y with domains and viewpoints that enable you to see the relationships between them.

13. $f(x, y) = x^2y^3$ Technology/black box stuff as time permits.

15–40 Find the first partial derivatives of the function.

15. $f(x, y) = x^4 + 5xy^3$

18. $f(x, t) = \sqrt{3x + 4t}$

21. $f(x, y) = \frac{x}{y}$

26. $u(r, \theta) = \sin(r \cos\theta)$

29. $F(x, y) = \int_y^x \cos(e^t)\, dt$

FTC I

30. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

47–50 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

y = constant in $\frac{\partial z}{\partial x}$ stuff

47. $x^2 + 2y^2 + 3z^2 = 1$

Assuming that z is implicitly a function of x & y.

$\frac{\partial z}{\partial x}$: $2x + 6z \cdot \frac{\partial z}{\partial x} = 0$

Chain Rule.

$6z z_x = -2x$

$z_x = \boxed{\frac{-2x}{6z} = \frac{\partial z}{\partial x}}$

50. $yz + x \ln y = z^2$

$\frac{\partial z}{\partial x}$: $y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$

$(y-2z)\frac{\partial z}{\partial x} = -\ln y$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\ln y}{2z-y}}$$

$\frac{\partial z}{\partial y}$: $\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$

$(1-2z)\frac{\partial z}{\partial y} = -\frac{x}{y}$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x}{y(2z-1)}}$$

51–52 Find $\partial z/\partial x$ and $\partial z/\partial y$.

52. (a) $z = f(x)g(y)$ (b) $z = f(xy)$
(c) $z = f(x/y)$

(a) $z = f(x)\,g(y)$

$$\boxed{\frac{\partial z}{\partial x} = \frac{df}{dx}\,g(y)} + f(x)\frac{\partial g}{\partial x} \to = 0$$

$\frac{\partial}{\partial x}\left[x^2 y^4\right] = 2xy^4$

$\frac{\partial z}{\partial y} = f(x)\frac{dg}{dy}$

53–58 Find all the second partial derivatives.

53. $f(x, y) = x^4y - 2x^3y^2$

56. $T = e^{-2r}\cos\theta$

$T_r = -2e^{-2r}\cos\theta$

$T_{rr} = 4e^{-2r}\cos\theta$

$T_{r\theta} = 2e^{-2r}\sin\theta$

$T_\theta = -e^{-2r}\sin\theta$

$T_{\theta\theta} = -e^{-2r}\cos\theta$

$T_{\theta r} = 2e^{-2r}\sin\theta$

59–62 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

59. $u = x^4y^3 - y^4$

$u_x = 4x^3y^3$ $\qquad$ $u_y = 3x^4y^2 - 4y^3$

$u_{xy} = 12x^3y^2 \quad = \quad u_{yx} = 12x^3y^2$ Same!

71. If $f(x, y, z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}.
[*Hint:* Which order of differentiation is easiest?]

$f_y = 2xyz^3 + 0$

$f_{yx} = 2yz^3$

$f_{yxz} = \boxed{6yz^2 = f_{xyz},}$ by Clairaut & continuity of everything in sight.

81. The *diffusion equation*

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi D t}}\,e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

83. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}$$

Find $\partial R/\partial R_1$.

$$R^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1}$$

$$\left(-R^{-2}\right)\frac{\partial R}{\partial R_1} = -R_1^{-2} + 0 + 0$$

$$\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2} = \frac{\left(\frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}\right)}{R_1^2} = \boxed{\frac{R_2R_3}{R_1(R_2R_3 + R_1R_3 + R_1R_2)}}$$

$$\frac{R_1R_2R_3}{R_1^2(\qquad)}$$