Chapter 13 Vector-valued functions.

$y = f(x)$

$\bar{F}(x) = \bar{F} = \langle x, f(x) \rangle$ has same graph, but is in vector form.

$\bar{r}(t) = \langle t, f(t) \rangle$

Generalize it:

$x = f(t)$, $y = g(t)$

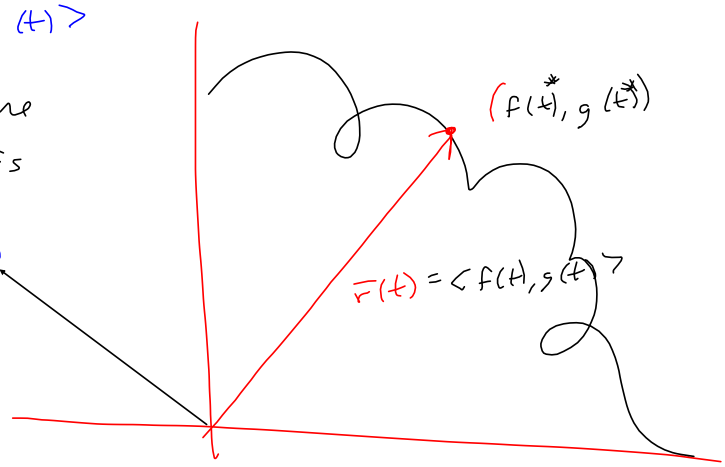
$\bar{r}(t) = \langle f(t), g(t) \rangle$

$= \langle x(t), y(t) \rangle$

Curvilinear

If $f(t)$ & $g(t)$ are functions, then so is $\bar{r}(t)$

No real vertical line test.

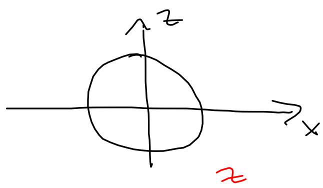


$\boxed{E}$ $\langle \sin(t), t, \cos(t) \rangle = \bar{r}(t)$

$x = \sin t$, $z = \cos t$

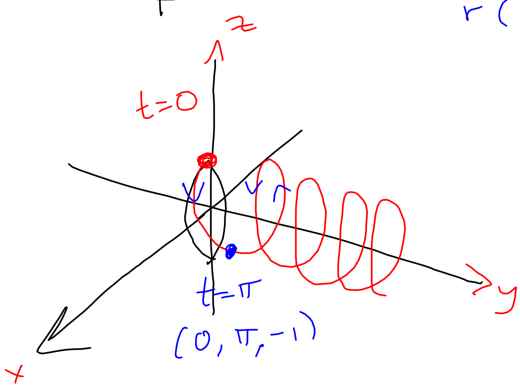
Then $x^2 + z^2 = 1$

Project into the xz-plane, it's a circle!



$\bar{r}(\pi) = \langle \sin\pi, \pi, \cos\pi \rangle$

$= \langle 0, \pi, -1 \rangle$



Derivatives & Integrals work just as you'd hope & expect

$$\int \bar{r}(t)\,dt = \int \langle x(t), y(t), z(t)\rangle\,dt$$

$$= \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

It all hinges on this notion:

$$\lim \langle a, b, c\rangle = \langle \lim a, \lim b, \lim c\rangle$$

Rules of differentiation:

$$\bar{r}'(t) = \frac{d}{dt}\bar{r}(t) = \frac{d\bar{r}}{dt} = \langle x', y', z'\rangle$$

$$(fg)' = \boxed{f'g + fg'}$$

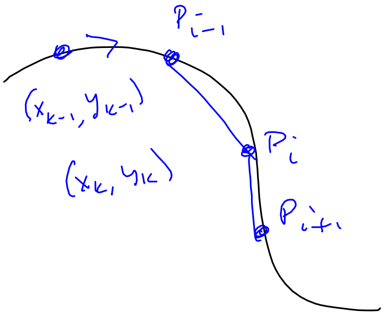
Let $\bar{u}(t)$ & $\bar{v}(t)$ be vector functions

Then $\frac{d}{dt}\left[\bar{u}(t)\cdot\bar{v}(t)\right] = \bar{u}'(t)\cdot\bar{v}(t) + \bar{u}(t)\cdot\bar{v}'(t)$

$$\frac{d}{dt}\left[\bar{u}(t)\times\bar{v}(t)\right] = \bar{u}'(t)\times\bar{v}(t) + \bar{u}(t)\times\bar{v}'(t)$$

$$= -\bar{v}(t)\times\bar{u}'(t)$$

Recall arc length



S = arc length

$$\approx \sum_{k=1}^{n} \|\overrightarrow{P_{k-1}P_k}\|$$

$$= \sum \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$= \sum \sqrt{\Delta x^2 + \Delta y^2} = \sum \sqrt{\left(\frac{\Delta x^2}{\Delta x^2} + \frac{\Delta y^2}{\Delta x^2}\right)(\Delta x)^2}$$

$$= \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x \xrightarrow[(\text{or } \Delta x \to 0)]{n \to \infty} \int_{x=a}^{x=b} \sqrt{1 + f'(x)^2}\, dx = S$$

Arc length for Space Curves

$$\ldots \sum \sqrt{\Delta x^2 + \Delta y^2} = \sum \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}$$

$$= \sum \sqrt{\left(\frac{(x(t_k) - x(t_{k-1}))^2}{(t_k - t_{k-1})^2} + \frac{(y(t_k) - y(t_{k-1}))^2}{(t - t_{k-1})^2}\right)(t_k - t_{k-1})^2}$$

$$= \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t$$

$$\xrightarrow{n \to \infty} \int \sqrt{x'(t)^2 + y'(t)^2}\, dt = \text{ARC LENGTH}$$

Arc length for vectors

$$\bar{r}(t) = \langle x(t), y(t) \rangle \Rightarrow \bar{r}'(t) = \langle x'(t), y'(t) \rangle$$

$$\Rightarrow \|\bar{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$$

$$\Rightarrow \text{ARC LENGTH}: \int \|\bar{r}'(t)\|\, dt \quad !$$

$\|\bar{r}'(t)\|\, dt = ds$ = increment of arc length

unit Tangent

$$\bar{T}(t) = \frac{1}{\|\bar{r}'(t)\|}\, \bar{r}'(t)$$