We have videos on 12.3 #s 36, 42. Check 'em out!

http://harryzaims.com/203/videos/chapter-12/12-03/

Recall $\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ = Determinant

$$\begin{array}{rl} & \langle a_1, a_2, a_3 \rangle, a_1, a_2 \\ \times & \langle b_1, b_2, b_3 \rangle, b_1, b_2 \\ \hline & \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \end{array}$$

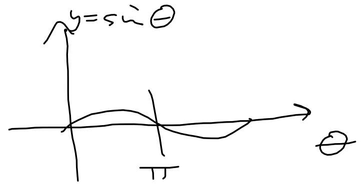
This is a VECTOR that's orthogonal (perpendicular) to **a** and **b**.

Right-hand-rule thing goin' on:

Curl fingers of right hand from **a** to **b** and your thumb will point in the direction of the cross product. "Tendency to rotate."

§12.4

NOTE: $0 \le \theta < \pi \Rightarrow \sin\theta \ge 0$.

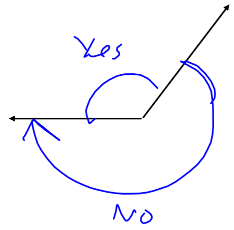


Theorem:

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$$

"Proof"

$$\|\vec{a}\times\vec{b}\|^2 = \|\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle\|^2$$



$$(x+y)^2 = x^2 + 2xy + y^2$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle \cdot \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2$$

$$+ a_3^2b_1^2 - 2a_3b_1a_1b_3 + a_1^2b_3^2$$

$$+ a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$$

& then, a miracle occurs.

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\|\vec{a}\|\|\vec{b}\|\cos\theta)^2$$

$$\|\vec{a}\|\|\vec{b}\|\cos\theta = \vec{a}\cdot\vec{b}$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2(1 - \cos^2\theta) = \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta = \|\vec{a}\times\vec{b}\|^2$$

$$= \sqrt{\|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta} = \sqrt{\|\vec{a}\times\vec{b}\|^2}$$

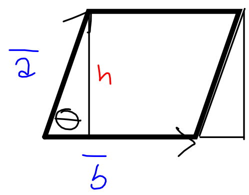
$$= \|\vec{a}\|\|\vec{b}\|\,|\sin\theta| = \|\vec{a}\times\vec{b}\|$$

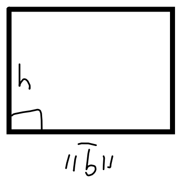
$$= \boxed{\|\vec{a}\|\|\vec{b}\|\sin\theta = \|\vec{a}\times\vec{b}\|}$$

(Drop the absolute value, b/c $0 \le \theta < \pi \Rightarrow \sin\theta \ge 0$)

$\boxed{\text{THEOREM}}$ $\|\bar{a} \times \bar{b}\|$ = the area of the parallelogram formed by $\bar{a}$ & $\bar{b}$

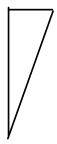
Area of rectangle





$$\frac{h}{\|\bar{a}\|} = \sin\theta \implies$$

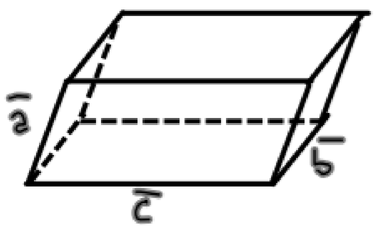
$$h = \|\bar{a}\| \sin\theta$$

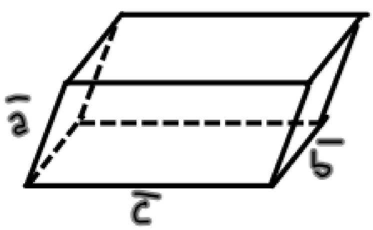


$$\text{Area} = h\|\bar{b}\| = \|\bar{a}\| \frac{\|\bar{a} \times \bar{b}\|}{\|\bar{a}\|\|\bar{b}\|} = \frac{\|\bar{a} \times \bar{b}\|}{\|\bar{b}\|}$$

$$= \frac{\|\bar{a} \times \bar{b}\|}{\|\bar{b}\|} \|\bar{b}\| = \|\bar{a} \times \bar{b}\|$$

$$\boxed{\text{Parallelogram Area} = \|\bar{a} \times \bar{b}\|}$$





FACT: Scalar Triple Product $= |\bar{a} \cdot (\bar{b} \times \bar{c})|$ = Volume of the parallepiped defined by $\bar{a}, \bar{b}, \bar{c}$.

§12.4 #4:

$\bar{j} + 7\bar{k} = \bar{a}, \quad 2\bar{i} - \bar{j} + 4\bar{k} = \bar{b}$

$4\bar{k} = 4\langle 0,0,1\rangle$

$\Rightarrow \bar{a} = \langle 0,1,7\rangle, \bar{b} = \langle 2,-1,4\rangle$

In higher mathematics, it's all column vectors.

$\bar{a} = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix} = 3 \times 1$ MATRIX (Linear Algebra, Advanced Calc, Diff Eq.)

$\bar{a} \times \bar{b}$:

$$\begin{array}{rl} & \langle 0, 1, 7\rangle, 0, 1 \\ \times & \langle 2, -1, 4\rangle, 2, -1 \\ \hline \end{array}$$

$\boxed{\bar{a} \times \bar{b} = \langle 11, 14, -2\rangle} = \bar{w}$

$\bar{a} \cdot \bar{w} = 0$ ✓

$\bar{b} \cdot \bar{w} = 0$ ✓

Student should provide more detail, for cogency.

S 12.4 Due Monday

13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ S

(b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ $\nexists$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ V

(d) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ $\nexists$

(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ $\nexists$

(f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ S

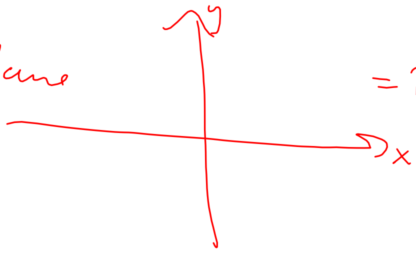
$D = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

Cartesian Product

$D: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$D(((1,2),(3,4))) = \sqrt{(1-3)^2 + (2-4)^2} = 2\sqrt{2} \in \mathbb{R}$

The plane



$= \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$