MAT 203-G11
Covers Chapter 14

Test 2 Take-Home – Spring, 2016
55 Points – 25% of Test 2 Grade

Name KEY

.Please, everyone follow the homework drill, except for folding. Plus EXTRA neatness and EXTRA space for max points.

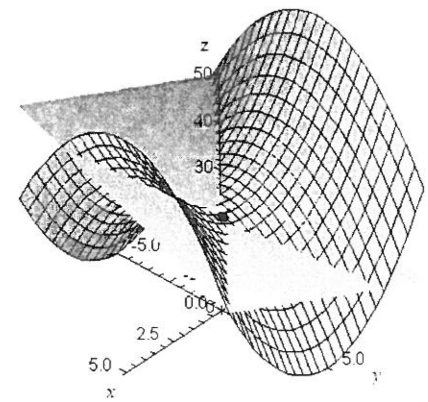
1. Answer one of the following:

   a. (5 pts) Find f_τ and f_ω if $f(\tau,\omega)=\int_{\sin(\tau)}^{\cos(\omega)}\pi\left(x^2e^{\pi x}\right)dx$.

   b. (5 pts) Find the 1st partials, f_x and f_y, for $f(x,y)=y\int_0^x \xi^3\arctan\left(\xi^4\right)d\xi$. FTC I with Product Rule!

2. (5 pts) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the equation: $yz+y\ln(x)=z^2$..

3. Let $f(x,y)=x^2-y^2+25$.



   a. (0 pts) Find an equation of the tangent plane to $f(x,y)$ at the point $(1, 1, 25)$. Get z all by itself in that equation, and you'll have the linearization, $z=L(x,y)$, that will give us the tangent plane approximations we ask for, next! :o)

   b. (0 pts) Use your previous answer to approximate $f(1.2, 1)$ and $f(1, 1.2)$. If you drew a blank on part 'a', make one up, as long as it contains $(1,1,25)$ and it's not horizontal or vertical.

   c. (0 pts) Find the *actual* values of $f(1.2, 1)$ and $f(1, 1.2)$.

   d. (5 pts) Find Δz for the change in z, from $f(1,1)=25$ to $f(1.2, 1)$ and then for the change in z from $f(1,1)=25$ to $f(1, 1.2)$. So you'll have two answers, right?

   e. (5 pts) Find the differential approximations, dz, to estimate the two answers in 'd' that you found, directly. Your answers to 'b' and $f(1,1)=25$ should come in, handy, here. Or just write the differential and plug in the relevant values of dx and dy in each expression. But you already have enough pieces to just subtract to the answer. Again, you'll have two answers.

   f. (5 pts) Which one of your answers in 'e' was an over-estimate, and which the under-estimate? Explain why, preferably using traces of the planes $x=k$ and $y=k$ in your reasoning. The picture can also be useful in your discussion.

4. Suppose the function $f(x,y)=x^2+4y^2-50$ describes an elliptical sinkhole in the middle of downtown Greeley (Pick your least favorite business.).

   a. (0 pts) What is the gradient at the point $(2,3,-10)$?

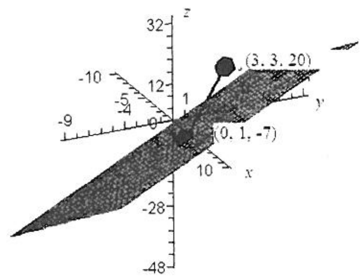
b. (0 pts) What is the directional derivative $D_{\overline{u}}$ in the direction of $\overline{u} = \langle 2,1 \rangle$?

c. (5 pts) Find a vector that points in the direction you would walk from the point $(2,3,-10)$ in order to stay on the $z=-10$ contour (level curve).

d. (5 pts) Sketch the $z=-10$ contour/level curve/intersection of z = -10 with the surface..

5. (5 pts) Find the distance from the point (3, 3, 20) to the plane $2x+2y-z=9$.

6. (5 pts) Find the point on the plane $2x+2y-z=9$ that's closest to (3, 3, 20).



7. (5 pts) Just in case you wanted to show me your triangle play, or you happen to have the derivative of $\arccos(x)$ handy, or can remember it:

Find the 1st partials, f_x and f_y, for $f(x,y) = \arctan(x^2y^2)$.

203 E2 JH

① (a) $f(z,\omega) = \int_{\sin z}^{\cos\omega} \pi x^2 e^{\pi x}\,dx$

$\Rightarrow f_z = \left(-\pi(\sin^2 z)e^{\pi \sin z}\right)(\cos z)$

& $f_\omega = \left(\pi(\cos^2\omega)e^{\pi\cos\omega}\right)(-\sin\omega)$

ANSWER ONE

5pts

(b) $f(x,y) = y\int_0^x \xi^3 \arctan(\xi^4)\,d\xi$

$\Rightarrow f_x = y\,x^3 \arctan(x^4)$

$f_y = \int_0^x \xi^3 \arctan(\xi^4)\,d\xi$

② $yz + y\ln x = z^2 \longrightarrow$

$yz' + \frac{y}{x} = 2zz'$

$yz' - 2zz' = -\frac{y}{x}$

$z'(y-2z) = -\frac{y}{x}$

$\boxed{z' = \frac{y}{x(2z-y)} = \frac{dz}{dx}}$

5pts

$z + yz' + \ln x = 2zz'$

$yz' - 2zz' = -z - \ln x$

$z'(y-2z) = -z - \ln x$

$\boxed{z' = \frac{dz}{dy} = \frac{\ln x + z}{2z-y}}$

203 E3 TH.

3 (a) $f(x,y) = x^2 - y^2 + 25$

Tan Plane @ $(1,1,25)$

$L(x,y) = f_x\,(x-1) + f_y\,(y-1) + 25$

$z = f(x,y) = x^2 - y^2 + 25 \Longrightarrow$

0 pts

$f_x = 2x$ & $\qquad f_y = -2y$

$f_x(1,1) = 2 \qquad f_y(1,1) = -2$

So, $\boxed{L(x,y) = 2(x-1) - 2(y-1) + 25}$

(b) $f(1.2,1) \approx L(1.2,1)$

$= 2(1.2-1) - 2(1-1) + 25$

0 pts

$= 2(.2) + 25$

$= \boxed{25.4 = L(1.2,1) \approx f(1.2,1)}$

$f(1,1.2) \approx L(1,1.2) = 2(1-1) - 2(1.2-1) + 25$

$= -2(.2) + 25$

$= -.4 + 25$

$= \boxed{24.6 = L(1,1.2) \approx f(1,1.2)}$

203 E2 P. H.

3c $f(1.2, 1) = (1.2)^2 - 1^2 + 25$

$= 1.44 - 1 + 25$

$= \boxed{25.44 = f(1.2, 1)}$

$f(1, 1.2) = 1^2 - (1.2)^2 + 25$

$= 1 - 1.44 + 25$

$= 25 - .44$

$= \boxed{24.56 = f(1, 1.2)}$

0pts

~~5pts~~

3d Find Δz

5pts

$f(1.2, 1) - f(1, 1) = 25.44 - 25 = \boxed{.44 = \Delta z}$

$f(1, 1.2) - f(1, 1) = 24.56 - 25$

$= \boxed{-.44 = \Delta z}$

3e $dz = L(1.2, 1) - f(1, 1) = \boxed{.4 = dz}$

5pts

$dz = L(1, 1.2) - f(1, 1) = \boxed{-.4 = dz}$

203 E2 TH

(3f) Along $x = 1$, the trace is concave down, so $L(1, 1.2)$ was over-estimate.

5pts

Along $y = 1$, the trace is concave up, and $L(1.2, 1)$ is an under-estimate

4d

(4a) $f(x,y) = x^2+4y^2-50$

0pts

$$\nabla f(2,3,-10) = \langle f_x, f_y \rangle = \langle 2x, 8y \rangle = \langle 4, 24 \rangle$$

(4b) $D_{\bar{u}}$ for $\bar{u} = \langle 2, 1 \rangle$

0pts

$$\|\bar{u}\| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$\nabla f(2,3) \cdot \frac{\bar{u}}{\|\bar{u}\|} = \langle 4, 24 \rangle \cdot \frac{1}{\sqrt{5}}\langle 2, 1 \rangle = \frac{1}{\sqrt{5}}(8+24) = \boxed{\frac{32}{\sqrt{5}} = D_{\bar{u}}}$$

OR $\frac{32\sqrt{5}}{5}$

(4c) To stay @ $z = -10$,

5pts

$z = x^2+4y^2-50$

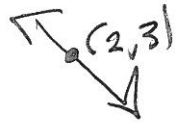
@ $z = -10$, we have

$x^2+y^2-50+10 = 0$

$x^2+y^2 = 40$

$2x + 8yy' = 0$

$y' = -\frac{x}{4y} = -\frac{2}{4(3)} = -\frac{1}{6}$



So, either $\langle 6, -1 \rangle$ OR $\langle -6, 1 \rangle$

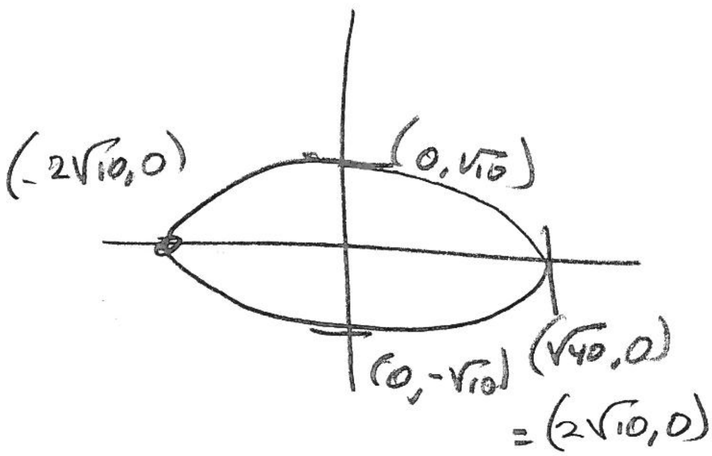
203 E2 TH

4d $x^2+4y^2-50=-10$

$$x^2+4y^2=40$$

$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

5pts



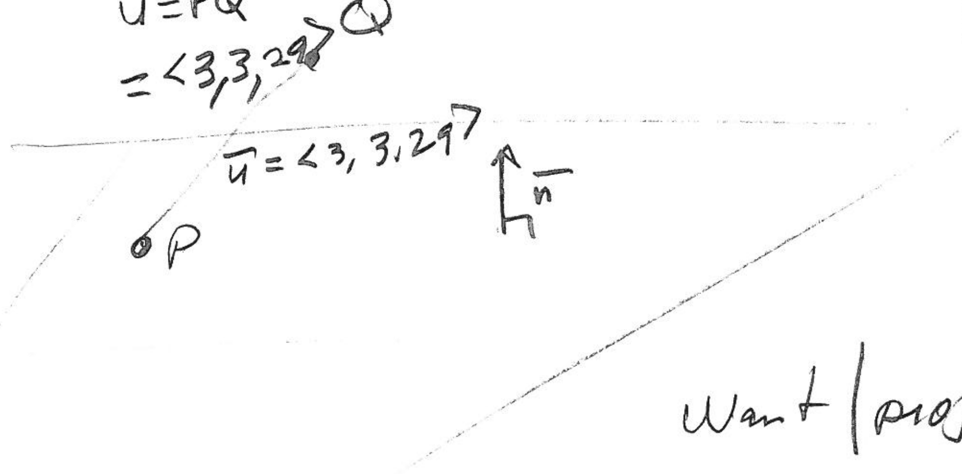
203 E2 TH

(5) Distance from $(3,3,20)$ to $2x+2y-z=9$

$Q(3,3,20)$ $\quad P(0,0,-9) \in P$

$\bar{u} = \overrightarrow{PQ}$
$= \langle 3,3,29 \rangle$

$\bar{n} = \langle 2,2,-1 \rangle$



$x=y=0$
$-z=9$
$z=-9$ ✓

Want $\left|\text{proj}_{\bar{n}} \bar{u}\right| =$

$$= \left|\text{comp}_{\bar{n}} \bar{u}\right| = \left|\frac{\bar{u}\cdot\bar{n}}{|\bar{n}|}\right|$$

$$= \quad = \left|\frac{\langle 3,3,29\rangle \cdot \langle 2,2,-1\rangle}{\sqrt{2^2+2^2+1^2}}\right|$$

5pts

$$= \frac{|6+6-29|}{\sqrt{9}} = \frac{|-17|}{3} = \boxed{\frac{17}{3}}$$

203 E2 TH

(6) We find closest point in the plane P to $(3,3,20)$.

Distance from (x,y,z) to $(3,3,20)$ is minimized when $f(x,y) = \text{distance}^2$ is minimized:

$$\begin{array}{r} 29 \\ 4 \\ \hline \end{array}$$

$$f(x,y) = (x-3)^2 + (y-3)^2 + (z-20)^2$$

$$= (x-3)^2 + (y-3)^2 + (2x+2y-9-20)^2$$

$$= (x-3)^2 + (y-3)^2 + (2x+2y-29)^2 \longrightarrow$$

$$\longrightarrow f_x = 2(x-3) + 2(2x+2y-29)(2)$$

$$= 2x-6+8x+8y-116$$

$$= 10x+8y-122 \overset{SET}{=} 0 \longrightarrow y = \frac{122-10x}{8}$$

$$f_y = 2(y-3) + 4(2x+2y-29)$$

$$= 2y-6+8x+8y-116$$

$$= 8x+10y-122 \overset{SET}{=} 0 \longrightarrow y = \frac{122-8x}{10}$$

$$\longrightarrow \frac{122-8x}{10} = \frac{122-10x}{8}$$

$$\longrightarrow 8(122) - 64x = 10(122) - 100x$$

$$\longrightarrow 36x = (10-8)(122) = 244$$

$$x = \frac{244}{36} = \frac{122}{18} = \boxed{\frac{61}{9} = x}$$

203 E2 TH

⑥ cnt'd

$$1220 - 122 = 1098, \quad 1098 - 610 = 488$$

$$\Rightarrow y = \frac{122-10x}{8} = \frac{122 - 10\left(\frac{61}{9}\right)}{8}$$

$$= \frac{122 - \frac{610}{9}}{8} = \frac{1098-610}{72} = \frac{488}{72}$$

$$= \frac{244}{36} = \frac{122}{18} = \boxed{\frac{61}{9} = y} \Rightarrow$$

$$244 - 81 = 163$$

$$\Rightarrow z = 2x + 2y - 9$$

$$= 2\left(\frac{61}{9}\right) + 2\left(\frac{61}{9}\right) - 9$$

$$= \frac{244}{9} - \frac{81}{9} = \boxed{\frac{163}{9} = z}$$

∴ The point in the ~~plane~~ P closest to $(3,3,20)$ is $\boxed{A = \left(\frac{61}{9}, \frac{61}{9}, \frac{163}{9}\right)}$

Check 2nd deriv test:

$f_{xx} = 10$, $f_{yy} = 10$, $f_{xy} = 8 = f_{yx}$ (Clairaut!)

$$\Rightarrow D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (10)(10) - (8)(8) < 0$$

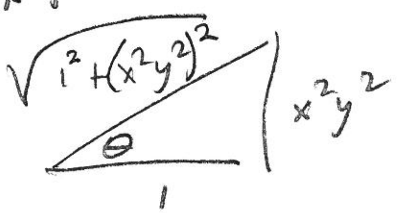
$\Rightarrow$ Min! ✓

203 E2 JH

(7) $f(x,y) = \arctan(x^2y^2)$

$$f_x = \frac{1}{f'(f^{-1}(x^2y^2))} = \frac{1}{\sec^2(\arctan(x^2y^2))} \cdot 2xy^2$$

$\theta = \arctan(x^2y^2)$



$$= \boxed{\frac{2xy^2}{(x^2y^2)^2+1} = f_x}$$

$$f_y = \frac{1}{f'(f^{-1}(x^2y^2))} = \boxed{\frac{2x^2y}{(x^2y^2)^2+1} = f_y}$$

5pts

$f(u) = \tan u$

$f'(u) = \sec^2 u$

$f^{-1}(u) = \arctan u$ $\qquad \dfrac{1}{f'(f^{-1}(u))} = \dfrac{1}{\sec^2(\arctan(u))}$

$u = x^2y^2$