MAT 203 Test 1 - Spring, 2016 Name KEY

100 Points Covers Chapters 12 and 13, Stewart's 7th Ed.

SUBMIT PROBLEMS ON SEPARATE PAPER. IN ORDER. FOLLOW HOMEWORK RULES (ONE-SIDE ONLY, MARGIN). EXCEPT DON'T FOLD YOUR TEST, OK? Remember, no double jeopardy. Any work that depends on previous work will be graded as though the previous work is correct. I'm afraid there's a lot of that on this test, but it should only hurt my weekend. LEAVING EXTRA ROOM AROUND YOUR WORK HELPS ME MAKE SURE YOU DON'T GET HURT TWICE FOR A PREVIOUS MISTAKE.

Work up to 2 Bonus problems, for up to 120 points.

1. (10 pts) Find parametric equations *and* vector equation for the line L that is the intersection of the two planes $\begin{matrix} P_1: & x - y + 3z = -2 \\ P_2: & 2x + 2y + 5z = 1 \end{matrix}$.

2. We're going to try to work our way up in dimensions, here, unlike the practice test.
   a. (10 pts) (Line) Let $A = (2,3,-1)$ and $B = (3,5,2)$. Form the vector $\overline{u} = \overrightarrow{AB}$, and write the vector equation for the line L containing A and B. If you were expecting parametric equations, go for that, then use your pametric equations to build the vector equation.
   b. (10 pts) (Line Segment) Write the equation for the line segment, from A to B.
   c. (10 pts) (Plane, Vector Equation) Let $C = (3,4,1)$. Form the vector $\overline{v} = \overrightarrow{AC}$, and write the vector equation for the plane containing A, B, and C. (Again, if you're geared-up for parametric equations, go for it.)
   d. **Bonus** (10 pts) (Plane, General Equation) Write the general equation for the plane P containing the 3 points, A, B and C.
   e. (10 pts) (Area of Parallelogram) Find the area of the parallelogram defined by the vectors $\overline{u}$ and $\overline{v}$.
   f. (10 pts) (Volume of a parallelepiped) Let $D = (4,6,3)$. Form the vector $\overline{w} = \overrightarrow{AD}$, and find the volume of the parallelepiped defined by $\overline{u}, \overline{v}$, and $\overline{w}$.

3. Distance questions.
   a. (10 pts) Find the distance from the point D to the line L.
   b. (10 pts) Find the distance from the point D to the plane P.
   c. **Bonus** (The one I can never do, but everyone else can.) (Distance between 2 planes) Take your answer from #2d, above, and I don't care *what* your a, b, c, and d are. But say your #2d was $P: ax + by + cz = d$. Add 40 to the right hand side, so you have a parallel plane $P_2: ax + by + cz = d + 40$. Find the distance between your P you built for #2d, and the P_2 you just built.

TEST 1

P_1: $x - y + 3z = -2 \Longrightarrow x = y - 3z - 2$

P_2: $2x + 2y + 5z = 1 \Longrightarrow 2(y - 3z - 2) + 2y + 5z = 1$

$\Longrightarrow 2y - 6z - 4 + 2y + 5z = 1$

$4y - 1z = 5$

$4y = z + 5$

$y = \dfrac{z+5}{4} = \dfrac{1}{4}z + \dfrac{5}{4}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -2 \\ 2 & 2 & 5 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -2 \\ 0 & 4 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{11}{4} & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{5}{4} \end{array}\right]$$

$x = \frac{11}{4}z - \frac{3}{4}$

$y = \frac{1}{4}z + \frac{5}{4}$

$z = z$

$x = y + 3z - 2 = \left(\frac{1}{4}z + \frac{5}{4}\right) - 3z - 2$

$= -\frac{11}{4}z - \frac{3}{4}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{4}z - \frac{3}{4} \\ \frac{1}{4}z + \frac{5}{4} \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{5}{4} \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}, \ z \in R$$

$\langle -\frac{3}{4}, \frac{5}{4}, 0 \rangle + t \langle -\frac{11}{4}, +\frac{1}{4}, 1 \rangle$

$\bar{r}(t) = \langle -\frac{3}{4}, \frac{5}{4}, 0 \rangle + t \langle -\frac{11}{4}, \frac{1}{4}, 1 \rangle$

$= \langle -\frac{3}{4}, \frac{5}{4}, 0 \rangle$

209 E1

(1) $A=(2,3,-1)$, $B=(3,5,2)$

$\overrightarrow{AB} = \bar{u} = \langle 1,2,3\rangle$

(a) $\boxed{\bar{r}(t) = \langle 2,3,-1\rangle + t\bar{u},\ t\in\mathbb{R}}$

$x = 2+t$
$y = 3+2t$
$z = -1+3t$

(b) $\boxed{\bar{r}(t) = (1-t)\langle 2,3,-1\rangle + t\langle 3,5,2\rangle,\ 0\le t\le 1}$

$= \langle 2-2t,\ 3-3t,\ -1+t\rangle + \langle 3t, 5t, 2t\rangle$

$= \langle 2+t,\ 3+2t,\ -1+3t\rangle$

(c) $C=(3,4,1) \longrightarrow \bar{v} = \overrightarrow{AC} = \langle 1,1,2\rangle$

$\boxed{\bar{r}(s,t) = \langle 2,3,-1\rangle + s\bar{u} + t\bar{v},\ (s,t)\in\mathbb{R}^2.}$ P

$= \langle 2,3,-1\rangle + s\langle 1,2,3\rangle + t\langle 1,1,2\rangle$

(d) $\bar{n} = \bar{u}\times\bar{v}$ is $\perp$ to plane.

P-vec $\langle 2,3,-1\rangle \in P$

$\langle 1,2,3\rangle$
$\times\ \langle 1,1,2\rangle$

$\langle 1, -(-1), -1\rangle$

$= \langle 1,1,-1\rangle = \bar{n}$

$\langle x,y,z\rangle \in P \longrightarrow$

$\bar{n}\cdot(\bar{x} - \langle 2,3,-1\rangle) = 0$

$1(x-2) + 1(y-3) - 1(z+1) = 0$

$x-2+y-3-z-1 = 0$

$x+y-z = 6$

303 E1

(2) (e) $\text{Area} = |\bar{u} \times \bar{v}| = |\bar{n}| = \sqrt{1^2+1^2+1^2} = \boxed{\sqrt{3}}$

$|\bar{u} \times \bar{v}| = |\bar{u}||\bar{v}|\sin\theta$

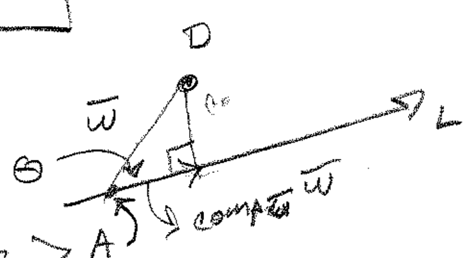
(f) $D = (4,6,3) \implies \bar{w} = \overrightarrow{AD} = \langle 2,3,4 \rangle$

$\text{Vol} = |\bar{w} \cdot (\bar{u} \times \bar{v})|$

$= |\bar{w} \cdot \bar{n}| = |\langle 2,3,4 \rangle \cdot \langle 1,1,-1 \rangle|$

$= |2+3-4| = \boxed{1}$

(3) Dist. from D to $\mathcal{L}$



$L: \langle 2,3,-1 \rangle + t \langle 1,2,3 \rangle$

$(2,3,-1) = A$

cosine leads me nowhere.

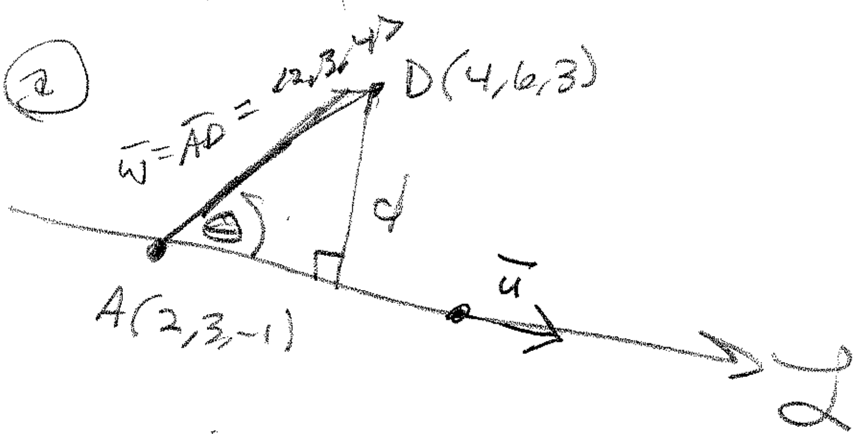
$\cos\theta = \dfrac{|\text{comp}_{\bar{u}} \bar{w}|}{|\bar{w}|} = \dfrac{\frac{|\bar{w} \cdot \bar{u}|}{|\bar{w}|}}{|\bar{w}|} = \dfrac{|\bar{w} \cdot \bar{u}|}{|\bar{w}|^2}$

$= \dfrac{\langle 2,3,4 \rangle \cdot \langle 1,2,3 \rangle}{2^2+3^2+4^2} = \dfrac{2+6+12}{4+9+16}$

$= \dfrac{20}{29} = \cos\theta \implies \theta = \cos^{-1}\left(\dfrac{20}{29}\right)$

203 TEST 1

3 (a)



$$\frac{d}{|\overline{w}|} = \sin\theta$$

$$d = |\overline{w}| \sin\theta = |\overline{w}| \frac{|\overline{w} \times \overline{u}|}{|\overline{w}||\overline{u}|} = \frac{|\overline{w} \times \overline{u}|}{|\overline{u}|}$$

$$= \qquad = \frac{\sqrt{1+4+1}}{\sqrt{1+4+9}} = \frac{\sqrt{6}}{\sqrt{14}} = \frac{\sqrt{3}}{\sqrt{7}}$$

$$= \boxed{\frac{\sqrt{21}}{7}}$$

$$\begin{array}{rl} \langle 2, 3, 4 \rangle & \overline{w} \\ \times\ \langle 1, 2, 3 \rangle & \times \overline{u} \\ \hline \langle 1, -2, 1 \rangle & = \overline{w} \times \overline{u} \end{array}$$

(I probly have it backwards, but we just need lengths, so OK.)

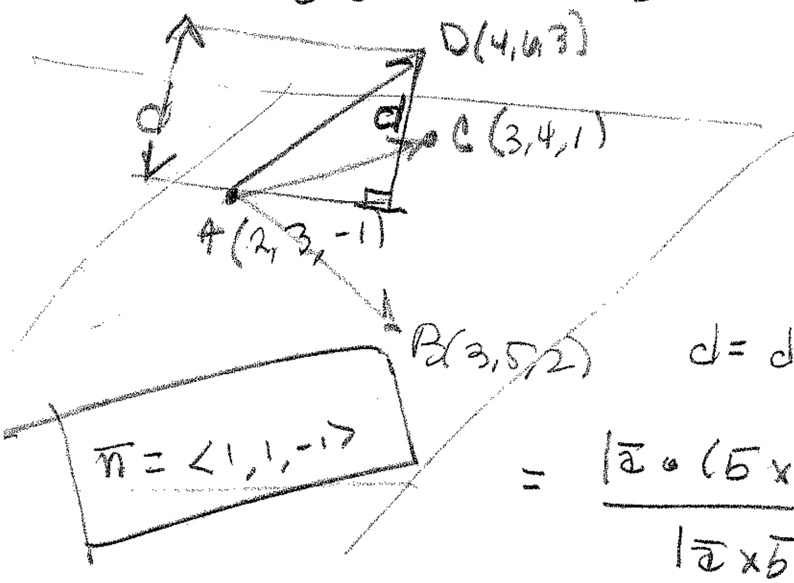
Is $\overline{B} = \overline{N} \times \overline{T}$ or $\overline{T} \times \overline{N}$?

pretty sure.

203 TEST 1

3b

dist from D to P



$\bar{u} = \overrightarrow{AB} = \langle 1, 2, 3 \rangle = \bar{a}$

$\bar{v} = \overrightarrow{AC} = \langle 1, 1, 2 \rangle = \bar{b}$

$\bar{w} = \overrightarrow{AD} = \langle 2, 3, 4 \rangle = \bar{c}$

$\boxed{\bar{n} = \langle 1, 1, -1 \rangle}$

d = distance

$$= \frac{|\bar{a} \cdot (\bar{b} \times \bar{c})|}{|\bar{a} \times \bar{b}|} = \frac{|\bar{u} \cdot (\bar{v} \times \bar{w})|}{|\bar{u} \times \bar{v}|}$$

$$= \frac{|(\bar{u} \times \bar{v}) \cdot \bar{w}|}{|\bar{n}|} = \frac{|\bar{n} \cdot \bar{w}|}{|\bar{n}|} = \frac{|\langle 1,1,-1 \rangle \cdot \langle 2,3,4 \rangle|}{\sqrt{1^2+1^2+(-1)^2}}$$

$$= \frac{|2+3-4|}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{3} = d}$$

(c) $x + y - z = 6$ Distance from this plane to this plane:

$x + y - z = 46$.

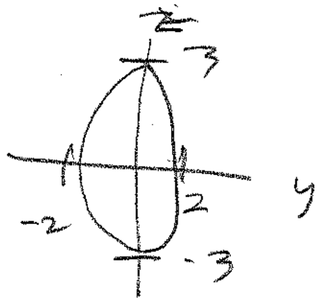
$$d = \frac{|d_1 - d_2|}{|\bar{n}|} = \frac{46-6}{\sqrt{3}} = \frac{40}{\sqrt{3}} = \boxed{\frac{40\sqrt{3}}{3}}$$

203 8 E1

(4) $9y^2+4z^2=36$

yz

$\frac{y^2}{4}+\frac{z^2}{9}=1$



xy

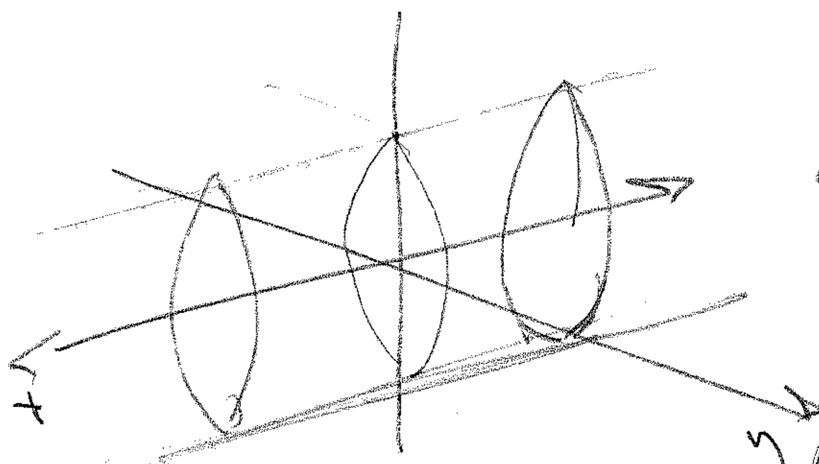
$9y^2=36$

$y^2=4$

$y=\pm 2$

xz

$z=\pm 3$



Elliptical cylinder,

Long axis = x-axis

Cross-sections are ellipses

Rulings ∥ to x-axis

203 E1

5

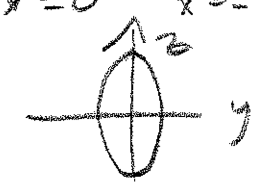
$$\frac{x^2}{16} = \frac{y^2}{9} + \frac{z^2}{25}$$

$x = 0$
$y = 0$
$z = 0$

$x = k$

$$\frac{y^2}{9} + \frac{z^2}{25} = \frac{k^2}{16}$$

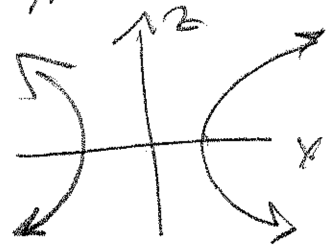
Ellipses
Long axis ∥ to
z-axis
Growing as x grows
Symmetric about
$x = 0$ $x = \pm k$



$y = \pm k$

$$\frac{x^2}{16} - \frac{z^2}{25} = \frac{k^2}{9}$$

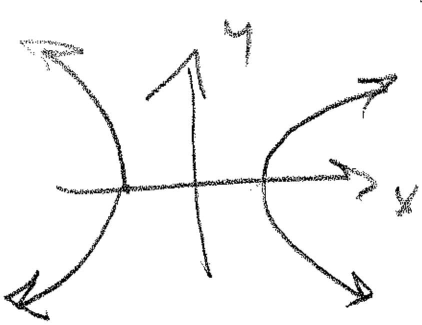
Hyperbolas



Elliptic
cone. Axis
∥ to x-axis

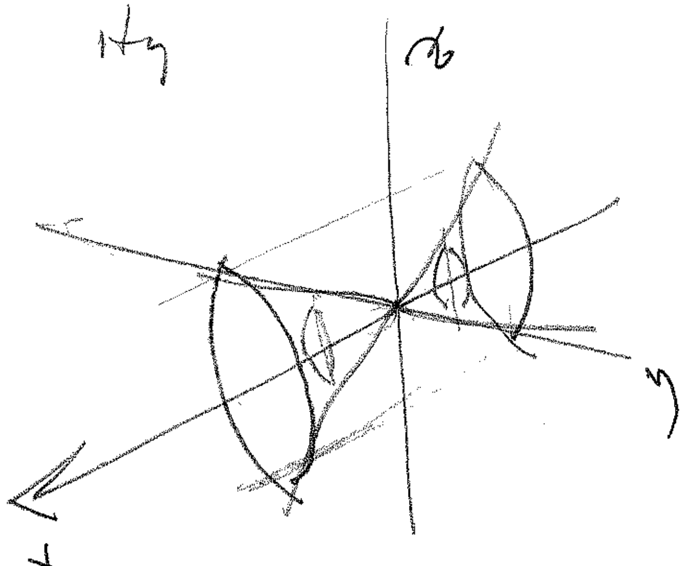
$z = \pm k$

$$\frac{x^2}{16} - \frac{y^2}{9} = \frac{k^2}{25}$$



$z = 0$ straight lines

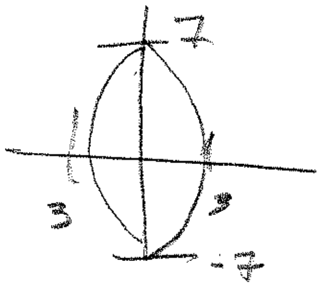
Hy



203 E1

(e) $\bar{r} = \langle 3\sin t, 7\cos t, t \rangle$

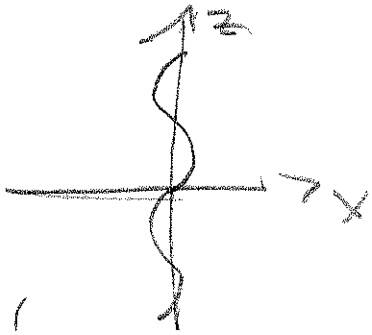
xy



$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$

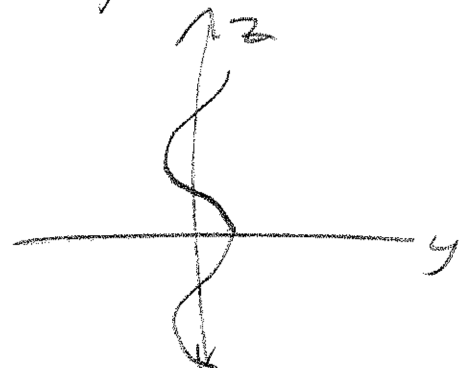
xz

$x = 3\sin z$

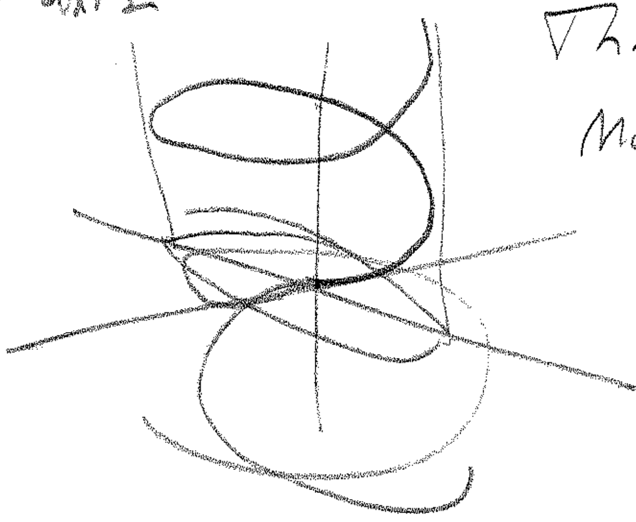


yz

$y = 7\cos z$



This is our buddy the barber pole, with elliptical cross-section long axis ∥ to z-axis



That's the idea.
More time = more refinements

203 E1

$\vec{r}(t) = \langle \sin(\frac{\pi}{3}t), \cos\frac{\pi}{3}t), 7 \rangle$

7

$\vec{r}'(t) = \langle \frac{\pi}{3}\cos\frac{\pi}{3}t, -\frac{\pi}{3}\sin\frac{\pi}{3}t, 0 \rangle$

$\vec{T}(t) = \frac{\vec{r}'}{|\vec{r}'|}$; $|\vec{r}'| = \sqrt{\frac{\pi^2}{9}(\cos^2 + \sin^2)}$

$= \frac{\pi}{3}$

$\vec{T} = \frac{3}{\pi}\langle \frac{\pi}{3}\cos\frac{\pi}{3}t, -\frac{\pi}{3}\sin\frac{\pi}{3}t, 0 \rangle = \langle \cos\frac{\pi}{3}t, \sin\frac{\pi}{3}t \rangle$

$= \langle \cos\frac{\pi}{3}t, -\sin\frac{\pi}{3}t, 0 \rangle$

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \frac{\langle -\frac{\pi^2}{9}\sin\frac{\pi}{3}t, -\frac{\pi^2}{9}\cos\frac{\pi}{3}t, 0 \rangle}{\sqrt{\frac{\pi^4}{9^2}(\sin^2\cancel{A} + \cos^2\cancel{A})}}$

$= \frac{9}{\pi^2}\langle -\frac{\pi^2}{9}\sin\frac{\pi}{3}t,$

Oh, I see. Duh.

$\vec{r}' = \frac{\pi}{3}\langle \cos\frac{\pi}{3}t, -\sin\frac{\pi}{3}t, 0 \rangle$

$|\vec{r}'| = \frac{\pi}{3}\sqrt{\cos^2\cancel{A} + \sin^2\cancel{A}} = \frac{\pi}{3}$ & so, $\cancel{\vec{T}}$ <

$\boxed{\vec{T} = \langle \cos\frac{\pi}{3}t, -\sin\frac{\pi}{3}t, 0 \rangle}$ & likewise

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \boxed{\langle -\sin\frac{\pi}{3}t, -\cos\frac{\pi}{3}t, 0 \rangle = \vec{N}}$

$\langle \cos\frac{\pi}{3}t, -\sin\frac{\pi}{3}t, 0 \rangle$

$\langle -\sin\frac{\pi}{3}t, -\cos\frac{\pi}{3}t, 0 \rangle$

$\langle 0, 0, -\cos^2\frac{\pi}{3}t - \sin^2\frac{\pi}{3}$

And $\vec{B} = \vec{T} \times \vec{N} =$

$= \boxed{\langle 0, 0, -1 \rangle = \vec{B}}$

Dot Product

$$a \cdot b = |a||b|\cos\theta$$

Cross-Product

$$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\theta$$

Scalar Projection

$$\text{Comp}_a b = \frac{a \cdot b}{|a|}$$

Vector Projection

$$\text{Proj}_a b = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|} = \frac{\bar{a} \cdot \bar{b}}{|a|^2} a$$

Where are the bars over the vectors?

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

Volume of a parallelopiped

$$V = |a \cdot (b \times c)|$$

Symmetric Equations

Solve parametric for t $\rightarrow$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Vector Equation of a Line

$$n \cdot r = n \cdot r_0$$

Scalar Equation of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Draw picture.

Distance From a Point to a Plane

$$D = |\text{comp}_n b| = \frac{|n \cdot b|}{|n|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Unit Tangent Vector

$$\bar{T}(t) = \frac{r'(t)}{|r'(t)|}$$

Acceleration

$$a = v' T + k v^2 N$$

Arc length

$$L = \int_a^b |r'(t)|\, dt$$

Torque

$$\tau = |r \times F| = |r||F|\sin\theta$$

Unit Normal Vector

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

Binomial Vector

$$B(t) = T(t) \times N(t)$$

Curvature

$$K_t = \left|\frac{dT}{ds}\right| = \left|\frac{T'(t)}{r'(t)}\right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$