§16.7 Surface Integrals

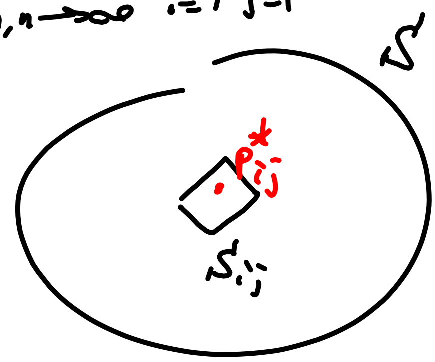
§16.7 #s 6, 15, 18, 24, 47, 49

#24 $\bar{n}$ is downward $= \langle 0, 0, -1 \rangle$

$\boxed{1}$ $\iint_S f(x,y,z)\, dS$ surface integral of f over S.

$$\lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

FIG 1



$\Delta S_{ij} = ?$

well,

$S = \{ \bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \mid (u,v) \in D \}$

We need continuity, differentiability of $\bar{r}$

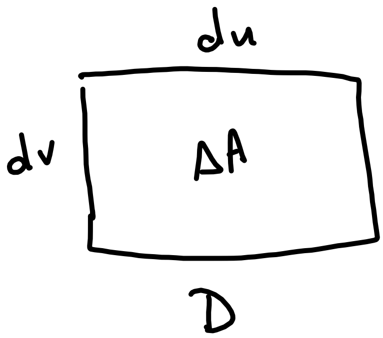
$\Delta S_{ij} = \| \bar{r}_u \times \bar{r}_v \| \Delta u \Delta v$

$\boxed{\text{AND}}$ need components $\neq 0$

and " not parallel.

Then if all is good

$\boxed{2}$ $\iint_D f(x,y,z)\, dS = \iint_D f(\bar{r}(u,v)) \| \bar{r}_u \times \bar{r}_v \|\, dA$



$\Delta A = \Delta u \Delta v \to du\,dv$

Line Integral

$$\int_C f(x,y)\,ds = \int_C f(\vec r(t))\,\overbrace{\|\vec r\,'(t)\|\,dt}^{ds}$$

$$f \equiv 1 \implies \iint_S f\,dS = \iint_D \|\vec r_u \times \vec r_v\|\,dA$$

$$= \text{Area of } S'$$

$$= A(S')$$

We compute surface integrals by converting it into a double integral over the parameter domain.

$\boxed{E1}$ $\iint_{S} x^2 \, dS$ where $S = \{(x,y) \mid x^2+y^2+z^2=1\}$

Spherical Coords : $x = \sin\phi\cos\theta$
$y = \sin\phi\sin\theta$
$z = \cos\phi$

So $\bar{r} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

Recall: $\|\bar{r}_u \times \bar{r}_v\| = \|\bar{r}_\phi \times \bar{r}_\theta\| = \sin\phi \quad (\rho = 1)$

$$\iint_S f \, dS = \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta \, \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos(2\theta)) \, d\theta \int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi$$

$$= \left[\int_0^{2\pi} \frac{1}{2} d\theta + \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \cos(2\theta)\,(2\,d\theta)\right]\left[\int_0^{\pi} \sin\phi\,d\phi + \int_0 \cos^2\phi \cdot (-\sin\phi\,d\phi)\right]$$

$$= \frac{1}{2}\theta\Big]_0^{2\pi} \quad = \cdots \quad = \frac{4\pi}{3} \; \blacksquare$$

Apps: $\iint_S \rho(x,y,z)\, dS$ Total Mass

$\iint_S q(x,y,z)\, dS$ Total Charge

$y = g(x,z)$?

Special Case: $z = g(x,y)$

$\bar{r} = \langle x, y, g(x,y)\rangle$ x & y parameters.

$\bar{r}_u = \bar{r}_x = \langle 1, 0, g_x \rangle$

$\times \quad \bar{r}_v = \bar{r}_y = \langle 0, 1, g_y \rangle$

$\langle -g_x, -g_y, 1 \rangle \Rightarrow$

$\|\bar{r}_x \times \bar{r}_y\| = \sqrt{g_x^2 + g_y^2 + 1}$

Arclength, $y = f(x) \rightarrow \bar{r} = \langle x, f(x) \rangle$

$\bar{r}' = \langle 1, f'(x) \rangle$

$\|\bar{r}'(t)\| = \sqrt{1 + (f'(x))^2}$

Book uses Liebniz notation, everywhere.

So $z = g(x,y) \Rightarrow$

$\boxed{4}$ $$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\, dA$$

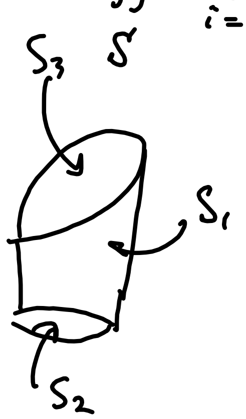
E 3 is Awesome

If $S = \bigcup_{i=1}^{n} S_i \Rightarrow \iint_S = \sum_{i=1}^{n} \iint_{S_i}$

S_1: $x^2+y^2=1$

S_2: $z = 0$

S_3: $z = x+1$



$\iint_S z\, dS$