Section 16.5 - Curl and Divergence

Here we try to get a way to quantify tendency to rotate, tendency to spread out, tendency to compress, etc. Here we get the 3-D description.

1 $$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Pretty intimidating, but you see y's and z's in the "i" or "x" spot, so you ought to be thinking cross product, and this is, indeed, a measure of "twist."

I'll show you the book way and then how I write it. You may use either one. Be careful about departing very far from both book and my way.

Let $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$

And when we "multiply times" a scalar function, the multiplication is actually the action of the partial derivative operators in the respective components of *f* .

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

It's so funny how they finesse multiplication into functions in operation, with the same notation. But they're trying to keep it from calculus and avoid all the machinery whose borders they're treading, to give us a practical grasp of what's happening.

Curl is the cross product of the dell operator with the vector function F-bar in the textbook's *ijk* notation. It's giving us a measure of the tendency to rotate. The first component is the system's tendency to twist in planes parallel to *x* = constant. The *j*-component is the tendency to twist in a plane parallel to *y* = constant.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \text{curl}\,\mathbf{F}$$

2 $$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

Don't this stuff make you think of that mysterious double integral in Green's Theorem? Wait for it....

3 **Theorem** If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Well of course it is! In a sense, we're taking the cross product of two vectors who are basically parallel. But it's a weird kind of parallel, by the weird kind of multiplication, defined by action of the differential operators on functions.

I always thought it was kind of ambiguous and ad-hoc, myself, because dell operator acts on real-valued functions of several variables AND it operates on VECTOR-valued functions in whatever way works to our advantage, apparently. Heh. Very beautiful, though, the way the language simplifies to the one powerful cross-product.

There's some awkwardness between my and the book's notation, because the subscripts for the order of the derivatives is the opposite (more algebraic) than the "operate from the left, only" action of the differential operators.

I'll try to remember to point this out, in class.

Book proof.

I think my vector and use of subscripts for partial derivatives is easier to work with, most of the way.

The *ijk* way can be helpful when you have a big problem and you can break it into the "i" problem, "j" problem, and "k" problem, and keep track.

PROOF We have

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

by Clairaut's Theorem. ■

Theorem 3 gives us a quick (or not) test for whether a vector field is conservative:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

Nice to know if it's a waste of time to try to find the function *f* for which F-bar is the gradient.

4 Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Ahhh. Now we have a sufficient condition. What we had before Theorem 4 was just a way to exclude. Now we have conditions to include. Good thing to have on a cheat sheet, I'd think.

I'd probably re-write my cheat sheet, and try to work things off it, as I became more able to write it all down, from an ideas point of view (ownership).

So far, it's all an "everywhere or nothing" in the conditions F-bar must satisfy. Wonder how much of this can all hold together, locally. Just because there are bad spots doesn't mean we have to go there, ya know?

Continuous partials, everywhere. How about continuous partials on some domain, if that domain's all we need?

Now, close your book (or at least no peeking) and see what you can do with this Example:

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

#37 is apparently a good problem to assign? I'm just gonna copy-and-paste the figure and the entire paragraph from the book

What is curl?

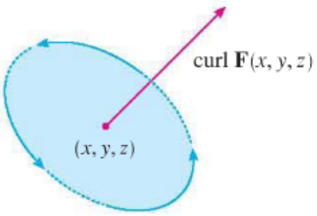

FIGURE 1

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 37. Another occurs when $\mathbf{F}$ represents the velocity field in fluid flow (see Example 3 in Section 16.1). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis. We give a more detailed explanation in Section 16.8 as a consequence of Stokes' Theorem.

Divergence. Hmmm. Looks like a dot product involving that pesky dell operator, again!

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Sho' 'nuff!

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

I like these proofs for students, because you just write what they tell you to write and you end with = 0.

Example 5 just seems like an exercise in the notation and meaning of the theorem. Not really something I've ever used, but be sure to tell me if you ever do.

If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Yet more notation abuse, for convenience. It's lasted because it works.

$$\nabla^2 = \nabla \cdot \nabla$$

This is called Laplace's operator, because of his equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

So it works this way, because we define it so. Easy to get tangled-up, but the context will usually carry you, as long as you don't stumble over the cross-products in Curl.

Vector Form of Green's Theorem.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

We use Curl and start getting scary echoes of Green's Theorem.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

12 $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA$$

The line integral on the left is the line integral of the tangential component of F-bar along C.

It's astounding that it's the double integral of the vertical (and only) component of the curl of F over the domain D enclosed by the scroc C.

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

Recall unit tangent vector from 13.2:

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

The OUTWARD unit normal is given by:

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

Why didn't they tell us? All this nonsense in Chapter 13 about the INWARD unit normal. Remember what that was?

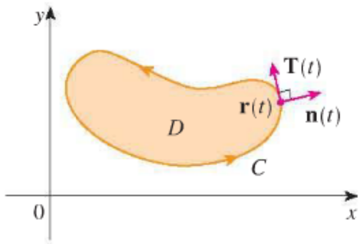

FIGURE 2

Put it together:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t), y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b P(x(t), y(t))\,y'(t)\,dt - Q(x(t), y(t))\,x'(t)\,dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA$$

The line integral of the *normal* component of the vector field F along the scroc C is the double integral of dell dot F-bar over the domain D that is enclosed by C.

That is, the line integral of the (outward) normal component of F-bar along scroc C is the double integral of div F-bar over the domain D.