Another one where the double beats the line...

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

OTOH, if you can recognize (or choose) a boundary over which P and Q are zero, then the double integral would collapse to zero, without your having to evaluate it.

Also, there's the connection between double integrals and area, when the overlying function is just z = 1.

So cook up a P and Q satisfying

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

P & $Q = 0$ on $C = \partial D$

$\Rightarrow \int_C P\,dx + Q\,dy = 0$ &

so $\iint_D (Q_x - P_y)\,dA = 0$

And turn the area question into a line integral.

Some suggested choices for P and Q with this property:

$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$

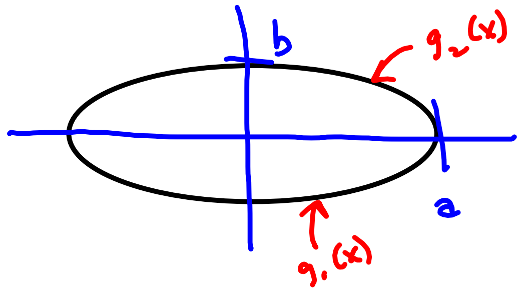
$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$

Then,

$$\boxed{5} \quad A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

3 different ways to express area of D when $\partial D = C$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



$$\int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} 1\,dy\,dx$$

$$= \int_{-a}^{a} (g_2(x) - g_1(x))\,dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$+ : g_2(x)$

$- : g_1(x)$

$P = -\frac{1}{2}y$

$Q = \frac{1}{2}x$

$Q_x - P_y = 1$

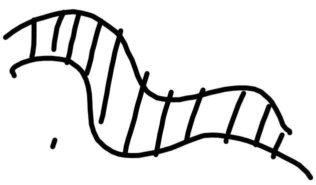
So $$\boxed{\int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} (Q_x - P_y)\,dA}$$

$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$

§16.4
HWs 1,3,5,8,11,14,17,19,27

$y = \pm b\sqrt{\frac{a^2-x^2}{a^2}}$

$= \pm \frac{b}{a}\sqrt{a^2-x^2}$



$$= \int_C P\,dx + Q\,dy$$

$$= \int_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$x = a\cos t \qquad dx = -a\sin t\,dt$

$y = b\sin t \qquad dy = b\cos t\,dt$

$$A = \frac{1}{2}\int_0^{2\pi} a\cos t\, b\cos t\,dt + b\sin t\, a\sin t\,dt$$

$$= \frac{1}{2}ab\int_0^{2\pi} dt = \frac{1}{2}ab\Big[t\Big]_0^{2\pi}$$

$$= \boxed{\pi ab}$$

Extending to more general domains, we just write them as the union of simple domains/regions and refer the reader back to the proof for simple regions.

Notice the extension to the union of two simple regions, and how you get paths that go in opposite directions and canceling-out. You see a lot of this kind of thing in topology.

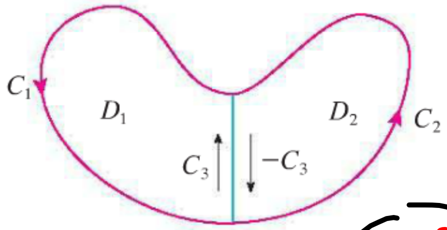
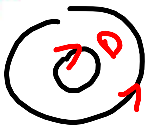


FIGURE 6

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

etc. Pic of a more complicated region, same idea. We're not too interested in that much rigor.

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Ugh! Looks like 4 line integrals! Maybe Green's can help!

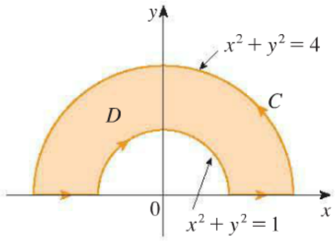


FIGURE 8

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA = \int_0^{\pi}\int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = [-\cos\theta]_0^{\pi}\left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$

$$3y - 2y = y = r\sin\theta$$

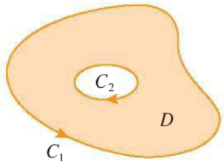


FIGURE 9

A few words about holes. Remember this: When making a circuit with your closed curves, always keep the region D on your left. Thus, the boundary of any holes inside of D, the boundaries are always traced CLOCKWISE when D is OUTSIDE the curve C.

Ordinarily, D is inside, and standard convention is COUNTERCLOCKWISE, with D on the inside.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

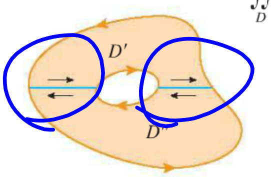


FIGURE 10

∂D = Boundary of D

= C

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$\vec{r}'(t)\,dt$

Key is it encloses the origin, so we know we can build an open disc inside it that also contains the origin.

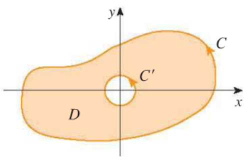


FIGURE 11

C' is shown with standard orientation, so when we are talking about D, its orientation will flipped, leading to the -C'.

Let's roll up our sleeves...

$\bar{F} = \langle -y, x \rangle$

$\boxed{Q_x = P_y} \Rightarrow$ Conservative?

or does Conservative $\Rightarrow Q_x = P_y$

$Q_x = 1,\ P_y = -1$

Not conservative.

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

$$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

Cool. So we now just have to evaluate the line integral about this nice circle!

$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j},\ 0 \le t \le 2\pi.$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$

$\bar{F} = \langle y, x \rangle$ satisfies

$Q_x = P_y = 1$, but <u>NOT</u>

conservative!

$\int_C \bar{F} \cdot d\bar{r} =$

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

$\int_c \bar{F} \cdot dr$ No. Too abstract.

30 minutes

$\boxed{6}$ **Theorem** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Curl & Divergence

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$- \bar{F} = \langle P, Q, R \rangle$

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle P, Q, R \rangle$$

$$\frac{\partial Q}{\partial y} = Q_y$$

Tendency to Rotate

$$\langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Green's Theorem.

$$\operatorname{div} \bar{F} = \nabla \cdot \bar{F}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= P_x + Q_y + R_z$$

S' 16.5 #s 1, 4, 7, 9-16, 21, 22