Section 16.4 - Green's Theorem

We survey the boundary to talk about what's inside the boundary.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

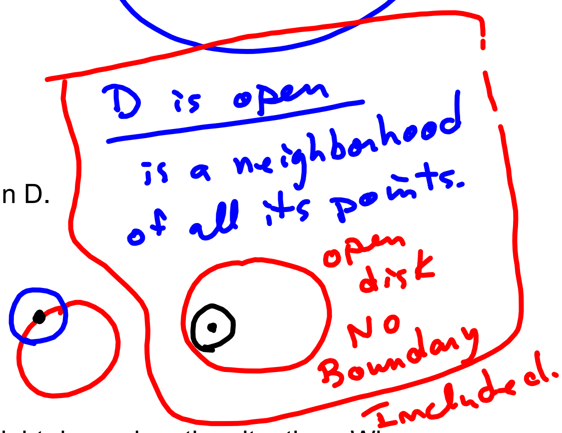
$D = (1,2)$ $\partial D = \text{Boundary} = \{1,2\}$

Notation: Positive Orientation conventions in the literature:

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \oint_C P\,dx + Q\,dy$$

"dell-d" for denoting the curve C as the boundary of a domain D.

1 $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$



One imagines that going one way or the other in equation 1 might depend on the situation. When you can recognize the one as the other, it might crack open a nasty-looking line integral or nasty looking double integral.

One also wants to remember everything pertinent to conservative vector fields, and to recognize when we're looking at a gradient in disguise, on the right! (When that happens, everything collapses to zero!).

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

→ FTC II
Net Change in $F(x)$

We're basically showing FTC II in reverse, with equation 1.

WTS:

2 *1st part of proof* *TYPE I*

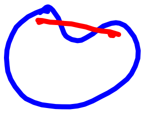
$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

3 *Type II*

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

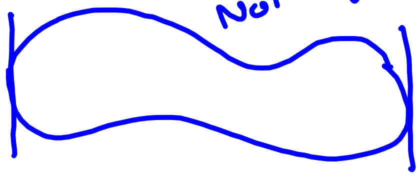
1

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

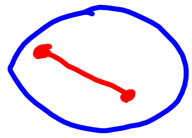


Not convex

Type I Not Type II



Convex set in plane



Line between 2 points stays inside

We basically assume D is a convex region (Type I and II), and leave the general prove to grad students.
We can at least get the general idea of how the argument needs to go.

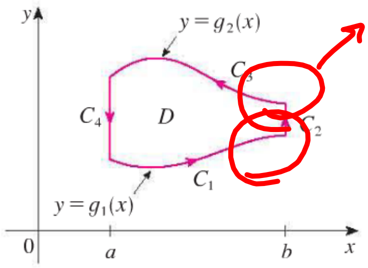
D as a Type I region. $D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$

4

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$

!

Note C3 and C1 are in opposite directions.



arguably concave, here, so NOT TYPE II

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

FIGURE 3

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx$$

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

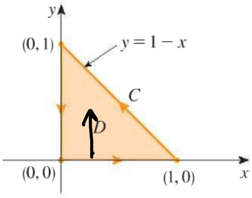


FIGURE 4

Here's one where the double integral over all of D is easier than writing 3 separate line integrals for 3 separate

$\int_C x^4\,dx + xy\,dy$ takes 3 separate line integrals.

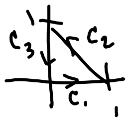
$$= \iint_D y\,dA = \int_0^1\int_0^{1-x} y\,dy\,dx \qquad u = 1-x,\ du = -dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{3}(1-x)^3\Big]_0^1 = -\frac{1}{6}[0-1] = \frac{1}{6}$$

$P = x^4 \qquad Q = xy$

$P_y = 0 \qquad Q_x = y$



C_1: $\vec{r} = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle$

$= \langle t, 0\rangle$

$x = t \qquad y = 0$

$dx = dt \qquad dy = 0$

$x = t$

$\frac{dx}{dt} = 1$

$dx = dt$

$$\oint_{C_1} P\,dx + Q\,dy = \int_{C_1} x^4\,dx + xy\,dy$$

$$= \int_0^1 t^4\,dt + \int_0^1 t\cdot 0\,dy = \frac{1}{5}$$

C_2: $\vec{r} = (1-t)\langle 1,0\rangle + t\langle 0,1\rangle$

$= \langle 1-t, t\rangle$

$x = 1-t \qquad y = t$

$dx = -dt \qquad dy = dt$

$\int (t - t^2) = \frac{1}{2}t^2 - \frac{1}{3}t^3$

$$\int_{C_2} = \int_0^1 (1-t)^4(-dt) + \int_0^1 (1-t)t\,dt$$

$$= + \left[+\frac{(1-t)^5}{5}\right]_0^1 + \left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1$$

$-\frac{1}{5} \qquad \frac{1}{6}$

$$= \boxed{-\frac{1}{30} \quad C_2}$$

C_3: $\vec{r} = (1-t)\langle 0,1\rangle + t\langle 0,0\rangle$

$= \langle 0, 1-t\rangle$

$x = 0 \qquad y = 1-t$

$dx = 0 \qquad dy = -dt$

$$\int_{C_3} x^4\,dx + xy\,dy = \int_0^1 0 + \int 0\cdot(1-t)(-dt)$$

$$= 0$$

So $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

$$= \frac{1}{5} - \frac{1}{30} = \boxed{\frac{1}{6}}$$

Another one where the double beats the line...

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right)dy$, where C is the circle $x^2 + y^2 = 9$.

OTOH, if you can recognize (or choose) a boundary over which P and Q are zero, then the double integral would collapse to zero, without your having to evaluate it.

Also, there's the connection between double integrals and area, when the overlying function is just z = 1.

So cook up a P and Q satisfying

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

P & Q = 0 on C = ∂D

$\Rightarrow \int_C P\,dx + Q\,dy = 0$ ∮

so $\iint_D (Q_x - P_y)\,dA = 0$

And turn the area question into a line integral.

Some suggested choices for P and Q with this property:

$P(x, y) = 0$ $\quad$ $P(x, y) = -y$ $\quad$ $P(x, y) = -\frac{1}{2}y$

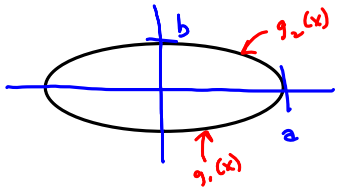
$Q(x, y) = x$ $\quad$ $Q(x, y) = 0$ $\quad$ $Q(x, y) = \frac{1}{2}x$

Then,

5 $$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

3 different ways to express area of D when ∂D = C

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



$$\int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} 1\,dy\,dx = \int_{-a}^{a} (g_2(x) - g_1(x))\,dx$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

+ : $g_2(x)$

− : $g_1(x)$

$P = -\frac{1}{2}y$

$Q = \frac{1}{2}x$

$Q_x - P_y = 1$

So $\int_{-a}^{a}\int_{g_1(x)}^{g_2(x)} (Q_x - P_y)\,dA$

$= \int_C P\,dx + Q\,dy$

$= \int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

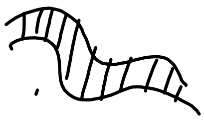
$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

§16.4 HWs 1, 3, 5, 8, 11, 14, 17, 19, 27

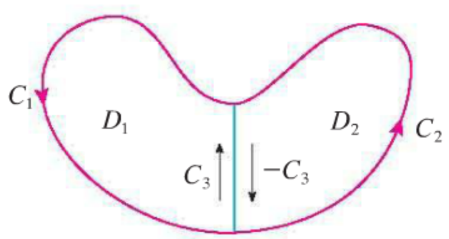
$y = \pm b\sqrt{\frac{a^2 - x^2}{a^2}}$

$= \pm \frac{b}{a}\sqrt{a^2 - x^2}$



Extending to more general domains, we just write them as the union of simple domains/regions and refer the reader back to the proof for simple regions.

Notice the extension to the union of two simple regions, and how you get paths that go in opposite directions and canceling-out. You see a lot of this kind of thing in topology.



$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

FIGURE 6

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

etc. Pic of a more complicated region, same idea. We're not too interested in that much rigor.

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Ugh! Looks like 4 line integrals! Maybe Green's can help!

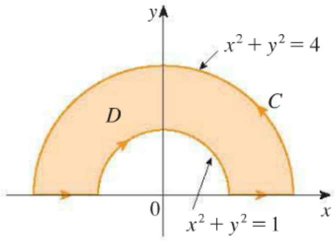


FIGURE 8

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA = \int_0^{\pi}\int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = [-\cos\theta]_0^{\pi}\left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$

$$3y - 2y = y = r\sin\theta$$

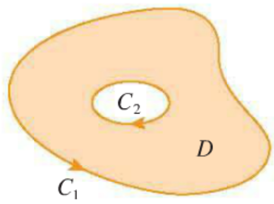


FIGURE 9

A few words about holes. Remember this: When making a circuit with your closed curves, always keep the region D on your left. Thus, the boundary of any holes inside of D, the boundaries are always traced CLOCKWISE when D is OUTSIDE the curve C.

Ordinarily, D is inside, and standard convention is COUNTERCLOCKWISE, with D on the inside.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

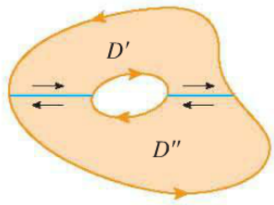


FIGURE 10

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Key is it encloses the origin, so we know we can build an open disc inside it that also contains the origin.

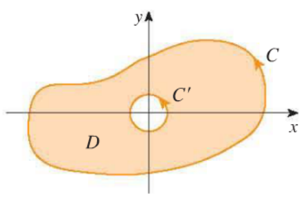


FIGURE 11

C' is shown with standard orientation, so when we are talking about D, its orientation will flipped, leading to the -C'.

Let's roll up our sleeves...

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

$$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

Cool. So we now just have to evaluate the line integral about this nice circle!

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j},\ 0 \leqslant t \leqslant 2\pi.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.