Let's set up time to sit down and render 3-D vector fields in my office.

§16.3 OLD-SCHOOL

Recall: FTC II:

$\boxed{1}$ $\int_a^b F'(x)\,dx = F(b) - F(a)$

$\boxed{T2}$ $F'(x)$ is now ∇f

C is a continuously diff'ble curve (smooth +)

f is " " func. of 2 vars.

$\Longrightarrow$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a)),$$ where

$C = \{ \bar{r}(t) \mid a \le t \le b \}$

when $\bar{F}$ is conservative $\int_C \bar{F} \cdot d\bar{r}$ is nice.

Line Integral of ∇f is the <u>net</u> <u>change</u> in f.

$$\int_a^b f'(x)\,dx = \underbrace{f(b)-f(a)}_{\text{Net change}}$$

Continuously Differentiable

-vs-

Differentiable

Advanced Calc is only place we care.

$$\int_C \nabla f \cdot d\bar{r} = \int_a^b \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle \, dt$$

$$= \boxed{\int_a^b (f_x x' + f_y y' + f_z z') \, dt}$$

$\bar{r} = \langle x, y, z \rangle$

$= \langle x(t), y(t), z(t) \rangle$

$\frac{d\bar{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$

$d\bar{r} = \langle x', y', z' \rangle dt$

Recall Chain Rule for these guys

$$\frac{d}{dt}(f(x(t), y(t), z(t)))$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= f_x x' + f_y y' + f_z z'$$

C is cnt$^{\underline{s}}$

$\bar{F} = \nabla f$ is conservative

$$\int_a^b f'(t) \, dt = \int_a^b \frac{df}{dt} \cdot dt$$

Big issue is the Domain has to be connected.

(Simply-connected, in the sequel)

$$= \int_a^b df = f \Big]_a^b =$$

$$= f(\bar{r}(b)) - f(\bar{r}(a)) \quad \blacksquare$$

Chain Rule version we used

§14.5 pg 948

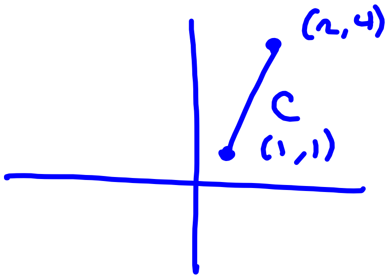
Pp 948-9 worth reviewing for that $\frac{df}{dt}$ thing.

$\varepsilon, \Delta x$ thing is more than we need,

$\boxed{E}$ Find work done by

$\bar{F} = \langle 2y^{\frac{3}{2}}, 3x\sqrt{y} \rangle$ moving an object

from $P(1,1)$ to $Q(2,4)$



$\bar{r}(t) = (1-t)\langle 1,1\rangle + t\langle 2,4\rangle$

$0 \le t \le 1$

$\bar{r}(0) = \langle 1,1\rangle$

$\bar{r}(1) = \langle 2,4\rangle$

FTC says we don't need this detail.

IF $\bar{F}$ is conservative.

means $\bar{F} = \nabla f$. we'll find f to prove it IS conservative & we'll need f to apply FTLI.

$\bar{F} = \langle 2y^{3/2}, 3x\sqrt{y} \rangle$ is, we hope

$\langle f_x, f_y \rangle$ for some f.

If it is, then

$$f = \int f_x\,dx = \int 2y^{3/2}dx = 2xy^{3/2} + \alpha(y)$$

$$\Rightarrow f_y = 3x\sqrt{y} = 3xy^{\frac{1}{2}} + \alpha'(y) \Rightarrow \int \alpha'(y) = \int 0$$

$\frac{d\alpha}{dy}$ $\Rightarrow \alpha(y) = C$

So, $f = 2xy^{\frac{3}{2}} + C$ $\Rightarrow$ $C = 0$ is easiest.

So FTLI, says

$$\int_C \bar{F} \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$$
$$= f(2,4) - f(1,1)$$
$$= 2(2)(4)^{\frac{3}{2}} - 2(1)(1)^{\frac{3}{2}}$$
$$= 32 - 2 = \boxed{30}$$

Summary of when $\bar{F}$ is conservative:

$\bar{F} = \langle P, Q \rangle = \langle f_x, f_y \rangle$

If $\bar{F} = \nabla f$ for some f, then

$P = f_x$, $Q = f_y$ & Clairaut

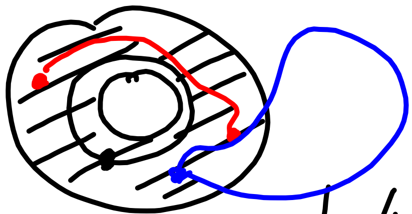
Says $f_{xy} = P_y = Q_x = f_{yx}$

Last one: $\bar{F} = \langle P, Q \rangle = \langle 2y^{\frac{3}{2}}, 3xy^{\frac{1}{2}} \rangle$

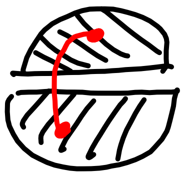
$$\Rightarrow \left.\begin{aligned} P_y &= f_{xy} = 3y^{\frac{1}{2}} \\ Q_x &= 3y^{\frac{1}{2}} \end{aligned}\right\} \text{Conservative!}$$

Simply Connected.

No Donuts

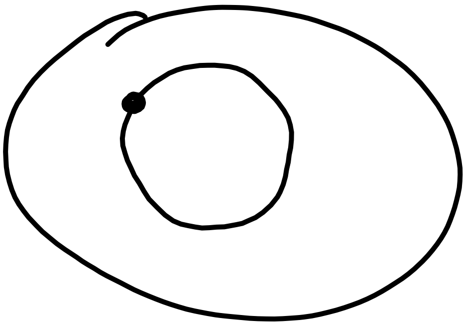


D is connected
but not simply
connected



Not connected

Connected - No separation
Simply connected - No holes.



$\boxed{E5}$ Finding f for 3-var case.

$\bar{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$

$f_x = y^2$

$\Rightarrow f = xy^2 + \alpha(y,z)$

$f_y = 2xy + \alpha_y(y,z) = 2xy + e^{3z}$

$\Rightarrow \alpha_y(y,z) = e^{3z}$

$\Rightarrow \alpha = ye^{3z} + \beta(z)$

$\Rightarrow f = xy^2 + \alpha(y,z) = xy^2 + ye^{3z} + \beta(z)$

$f_z = 3ye^{3z} + \beta_z(z) = 3ye^{3z}$

$\Rightarrow \beta_z(z) = 0 \Rightarrow \beta = K$

$f = xy^2 + ye^{3z} + K$

Take $K=0$ in apps.