Section 16.2 Line Integrals
Invented in the 19th century, motivated by fluid flow, force, electricity and magnetism.

"Hello!" said the dragon.

$\boxed{1}$ $x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = \langle x(t), y(t) \rangle$

As usual, divide $[a,b]$ into n equal-width subintervals. $\Delta t = \frac{b-a}{n}$. Then look at the corresponding arc length increment Δs. And ds is our old arc length friend

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ , and so}$$

$\boxed{2}$ **Definition** If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \boxed{3}$$

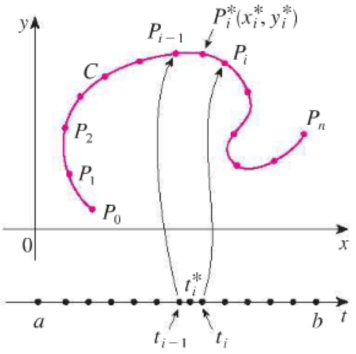


FIGURE 1

An *excellent* visual on what we're looking at. It's basically the signed area "under" f "above" the curve C. Shown, below, is the picture for a curve C that lives entirely above the xy-plane. In full-on general situation, the curve C lives in 3-D, and f is a number (z-value) assigned to the projection of C in the xy-plane. That'd complicate the picture, quite a bit, especially if C contains any loops! But the integral still works, and the basic idea is given by the special case where $y = 0$ everywhere along C.

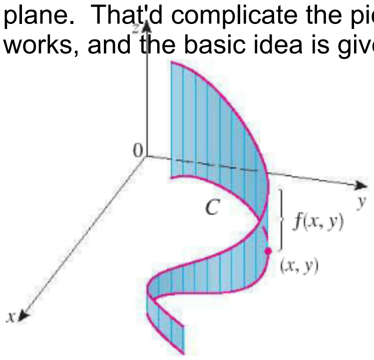


FIGURE 2

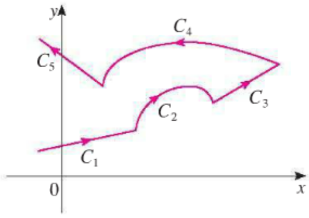


FIGURE 4
A piecewise-smooth curve

A piecewise-smooth curve, we break up [a, b] into subintervals on which C is smooth and use however many intervals we require to evaluate.

$$\int_C f(x, y)\,ds = \int_{C_1} f(x, y)\,ds + \int_{C_2} f(x, y)\,ds + \cdots + \int_{C_n} f(x, y)\,ds$$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

C_1: $x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$ $\qquad x' = \frac{dx}{dt}, \; y' = \frac{dy}{dt}$

x is the parameter.

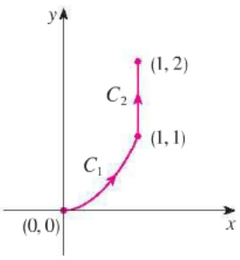


FIGURE 5
$C = C_1 \cup C_2$

$$\int_0^1 2x\sqrt{1+4x^2}\,dx \qquad ds = \sqrt{1+(y')^2}\,dx$$

$$= \frac{1}{4}\int_0^1 \sqrt{4x^2+1}\cdot 8x\,dx \qquad u = 4x^2+1, \quad du = 8x\,dx$$

$x = 1$

$$= \frac{2}{3}\cdot\frac{1}{4}\left(4x^2+1\right)^{\frac{3}{2}}\Big]_0^1$$

$$= \frac{1}{6}\left[5^{3/2} - 1\right]$$

C_2: $x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$ $\qquad ds =$

$$\int_{C_2} 2x\,ds = \int_0^1 2\,dy = 2y\Big]_0^1 =$$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5}-1}{6} + 2$$

$f(x,y) = \text{density} \Rightarrow \int f\,ds = \text{mass of wire.}$

Center of mass:

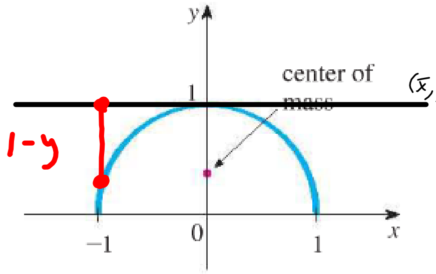
$\boxed{4}$ $\bar{x} = \frac{1}{m}\int x f\,ds$, $\bar{y} = \frac{1}{m}\int y f\,ds$ $\qquad (\bar{x},\bar{y}) = \text{C.o.m.}$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

All point@ on below $y=1 \Rightarrow 1-y$ is distance from (x,y) to $y=1$.

More interested in setup than finishing this in class.

$f(x,y) = k(1-y)$



$(\bar{x},\bar{y}) = \left(0, \frac{4-\pi}{2(\pi-2)}\right) \approx (0, 0.38)$

FIGURE 6

$$\int_C k(1-y)\,ds$$

$C: \quad x^2+y^2=1$

$x = r\cos\theta \qquad y = \sin\theta$

$= \cos\theta$

$$ds = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}\,d\theta$$

$$= d\theta$$

$$k\int_C (1-y)\,ds = 2k\int_0^{\frac{\pi}{2}} (1-\sin\theta)\,d\theta$$

$$= 2k\left[\theta + \cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2k\frac{\pi}{2} + 2k\cos\frac{\pi}{2} - 2k$$

$$= k\pi - 2k$$

Line integrals with respect to x or y:

$$\boxed{5} \quad \int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

$$\boxed{6} \quad \int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

Now, if $x = x(t)$, then $dx = x'(t)\,dt$

$y = y(t) \Rightarrow dy = y'(t)\,dt$, and $\boxed{5}$ & $\boxed{6}$ may be written:

$$\boxed{7} \quad \int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Sometimes you'll see them, together, in which case the conventions for writing them really suck:

$$\underbrace{\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy}_{\text{No prob.}} = \underbrace{\int_C P(x, y)\, dx + Q(x, y)\, dy}_{\text{GAG}}$$

Vector notation can be very nice, when trying to parametrize a curve. For instance, the following is a great way to think of a line segment.

The line segment $\bar{r}$, from $\bar{r}_0$ to $\bar{r}_1$:

$$\boxed{8} \quad \mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

$t = 0$: $\bar{r}(t) = \bar{r}_0$

$t = 1$: $\bar{r}(t) = \bar{r}_1$

Tidy.

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

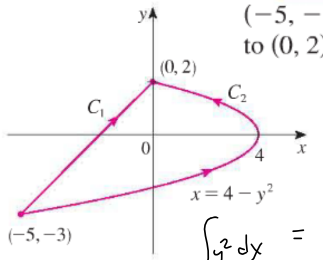


FIGURE 7

$$\int_C y^2 dx + \int_C x\,dy$$

C_1: Line segment

$$\vec r(t) = (1-t)\langle -5, -3\rangle + t\langle 0, 2\rangle$$
$$= \langle 5t-5, -3+3t\rangle + \langle 0, 2t\rangle$$
$$= \langle 5t-5, 5t-3\rangle$$

$x = 5t-5$, $y = 5t-3$

$dx = 5\,dt$ $dy = 5\,dt$

C_2: $x = 4 - y^2$ $y = y$

$(\vec r = \langle 4-y^2, y\rangle)$

$x = 4-y^2$ $y = y$

$dx = -2y\,dy$ $dy = dy$

$$\int_C y^2 dx = \int_{C_1} y^2 dx + \int_{C_2} y^2 dx$$
$$= \int_0^1 (5t-3)^2 (5\,dt) + \int_{-3}^{2} y^2(-2y\,dy)$$
$$= \frac{35}{3} + \frac{65}{2}$$

$$\int_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy =$$
$$= \int_0^1 (5t-5)(5\,dt) + \int_{-3}^{2} (4-y^2)\,dy$$
$$= -\frac{25}{2} + \frac{25}{3}$$
$$= -\frac{25}{6}$$

NOTE: $\int_{C_1} = \frac{35}{3} - \frac{25}{2} = \frac{70-75}{6} = -\frac{5}{6}$

$\int_{C_2} = \frac{65}{2} + \frac{25}{3} = \frac{195+50}{6} = \frac{245}{6}$

C_1 & C_2 start & end @ the same point, but $\int_{C_1} \neq \int_{C_2}$. NOT independent of path. (§16.3 has some conditions under which line integral is independent of path.)

$$\int_C f\,dx = -\int_{-C} f\,dx \quad , \quad \int_C f\,dy = -\int_{-C} f\,dy \quad , \underline{\text{but}}$$

$\int_C f\,ds = \int_{-C} f\,ds$, b/c the increment $\Delta s \geq 0$, but dx, dy

change sign when we reverse direction.

3-D!

$\boxed{9}$ $$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_a^b \bar{r}(t) |\bar{r}'(t)|\,dt \quad , \quad \text{as } ds = |\bar{r}'(t)|\,dt$$

And line integral wrt works just like wrt x & y

$z = z(t)$

$\frac{dz}{dt} = z'(t) \implies dz = z'(t)\,dt$

$\bar{r} = \langle x, y, z \rangle$

$\bar{r}' = \langle x', y', z' \rangle$

$|\bar{r}'| = \sqrt{x'^2 + y'^2 + z'^2}$

$\boxed{10}$ Ugh $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

I hate the notation & it always confused me as a student

Line Integrals of Vector Fields

Work = (Force)(Distance) = $\bar{F} \cdot \bar{D}$

$$\bar{T} = \frac{\bar{r}'}{|\bar{r}'|}$$

$\vec{PQ} = \bar{D}$

Now, $\bar{F} = \langle P, Q, R \rangle$

Over short distance: $W_i = \bar{F} \cdot \left[\overbrace{\Delta s_i}^{\text{Length}} \, \overbrace{\bar{T}(t_i)}^{\text{Direction}} \right]$

Add 'em up: $\sum \bar{F} \cdot \bar{T} \Delta s \xrightarrow{n \to \infty} \boxed{\overbrace{\int_C \bar{F} \cdot \bar{T} \, ds}^{\text{work}} = W \quad \boxed{12}}$

$$= \int_a^b \left[\bar{F}(\bar{r}(t)) \cdot \frac{\bar{r}'(t)}{|\bar{r}(t)|} \right] |\bar{r}'(t)| \, dt = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) \, dt$$

Summary:

13 Definition Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

$$ds = \sqrt{x'^2 + y'^2 + z'^2} \, dt$$

$$= |\bar{r}'| dt$$

And here's why they like crap notation Pg 1096

$\bar{F} = \langle P, Q, R \rangle$

$$\int_c \bar{F} \cdot dr = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt = \int_a^b \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle \, dt$$

$$= \int_a^b (Px' + Qy' + Rz')\, dt = \int_a^b Px'dt + \int_a^b Qy'dt + \int_a^b Rz'dt$$

$$= \int_a^b P dx + \int_a^b Q dy + \int_a^b R dz$$

$$= \text{UGH!} = \int_a^b P dx + Q dy + R dz$$

which is crap.

But remember it, because it's how the jerks like writing it.