Section 16.1 - Vector Fields

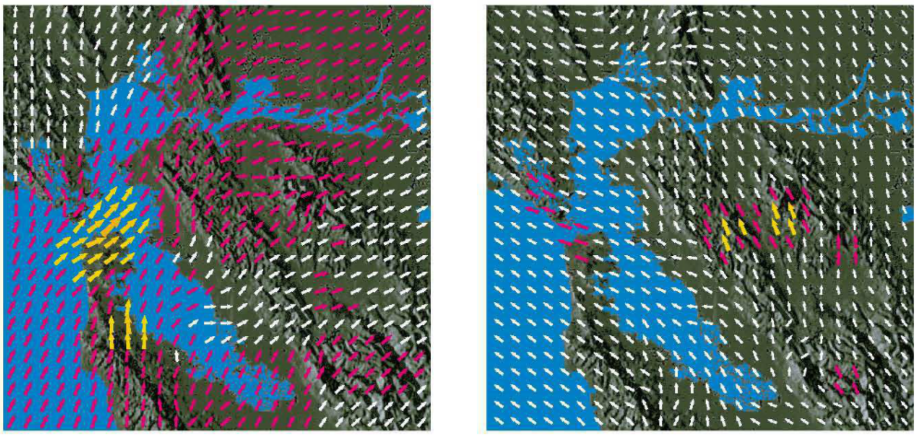
S 16.1 #s 1, 4, 6, 15 - 18*, 20*, 21 - 24, 35

*#s 15 - 18, 20 - WolframAlpha seems capable of rendering these things. YOUR job will be to find some way to convert your work to a PDF. I don't want you e-mailing me a bunch of image files.

$\boxed{1}$ **Definition** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

As we learn this stuff, keep in mind how clumsy the machinery is, compared to your intuitive grasp of wind velocity maps, ocean current maps, gravity field maps. We're so good at dealing with these beasts, empirically, that when it comes down to it, in real life, you might just use a million sensors and be taking a million measurements and letting a computer render the raw data for you.

But looking at wind maps, with those vertical torque vectors, especially if you can see one over a cyclone or a tornado, where the torque vectors stack up like a volcano!



(a) 6:00 PM, March 1, 2010 (b) 6:00 AM, March 1, 2010

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns

Length and color-coding to indicate velocity. But the vector field, expressed in this way, is easy to interpret. We know which way the wind's blowing, at a glance, above, or which way the ocean currents are flowing, with a glance at the next two images, stolen from your textbook.

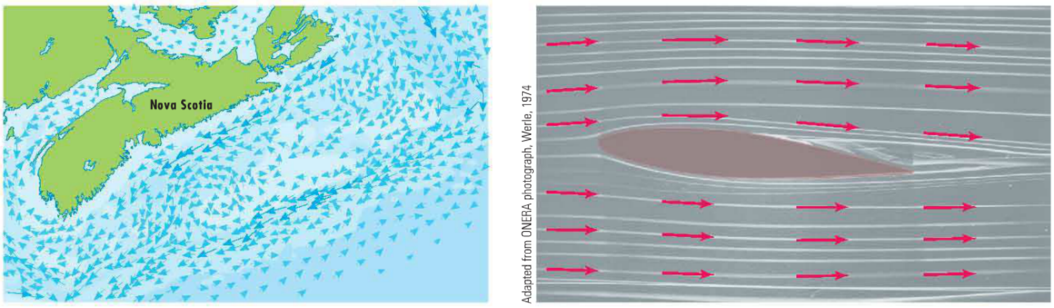
$$f(\tau, \omega) = \int_{\tan \tau}^{\omega^2} \frac{2}{x^2+1}\,dx$$

$$f_\tau = -\left(\frac{2}{\tan^2 \tau}\right)(\sec^2 \tau)$$

$$f_\omega = \left(\frac{2}{\omega^4+1}\right)(2\omega)$$

$$\frac{df}{d(\omega^2)} \qquad \frac{d(\omega^2)}{d\omega}$$

Ocean currents off Nova Scotia, on the left. Airflow past a wing. Easy 2 c what's going on.



(a) Ocean currents off the coast of Nova Scotia

(b) Airflow past an inclined airfoil

FIGURE 2 Velocity vector fields

Gravitational field is arrows pointed toward the origin (or center of the gravity sink).

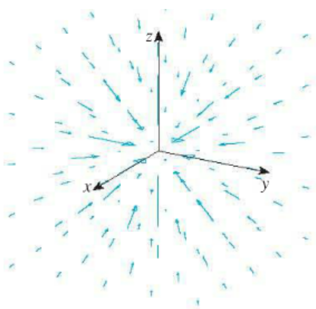


FIGURE 14
Gravitational force field

Accompanies Example 4.

Gravity,
magnetism
sound
electrostatic.

Conservative
fields

2 **Definition** Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

We will try to improve the clumsiness of Stewart's formalities, with some solid semi-formalities.

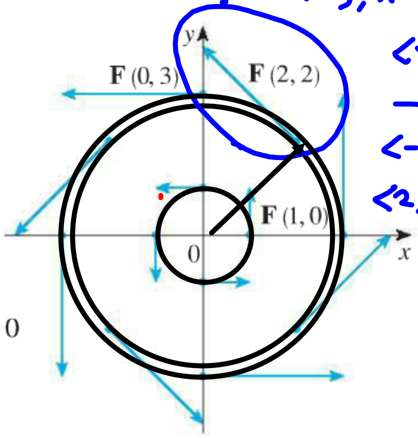
$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

$$= \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle = \langle P, Q, R \rangle = \vec{F}$$

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = \boxed{-y\,\mathbf{i} + x\,\mathbf{j}}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

$\vec{F} = \langle -y, x \rangle$

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$



$\langle x, y \rangle \longrightarrow \langle -y, x \rangle$

$\langle 2,2 \rangle = \langle -2, 2 \rangle = \vec{F}(2,2)$

FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + yx = 0$$

$$\langle x, y \rangle \cdot \langle -y, x \rangle = 0$$

Magnitude of the image

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

If you think of F as a mapping from R2 to R2, then it is apparently distance-preserving.

Distance-/Metric-/Norm-preserving mappings are special.

Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.

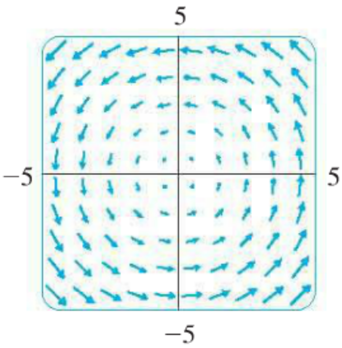
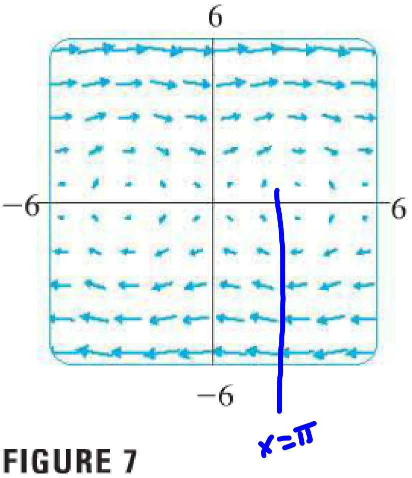


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$



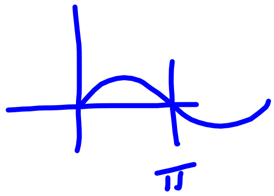


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

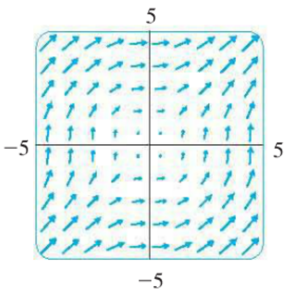


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

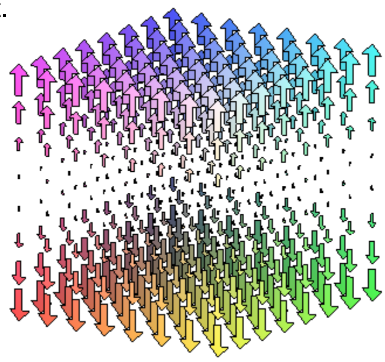
$with(plots)$ Maple can be very useful, here.

$with(VectorCalculus):$

$fieldplot3d(\langle 0, 0, z\rangle, x = -5..5, y = -5..5, z = -5..5, arrows = THICK)$

This is the field for Example 2, in the text.

$$\mathbf{F}(x, y, z) = z\,\mathbf{k}.$$



These are about as good as it gets in a static, monochrome medium. You really want these things "live" in a CAS session, so you can manipulate them in realtime. It's easy to misinterpret depth and direction in these 2-D projections of these vector fields in 3-D space.

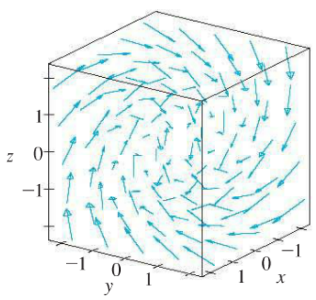


FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

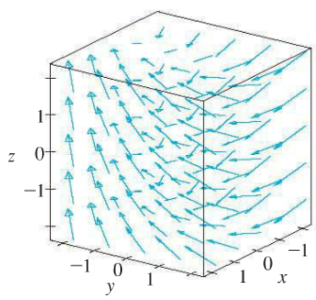


FIGURE 11
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

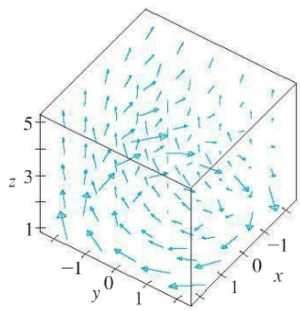


FIGURE 12
$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$

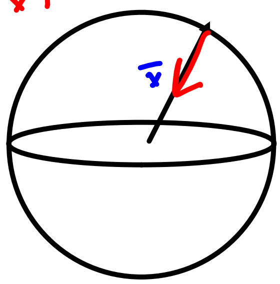
EXAMPLE 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = -\frac{mMG}{r^2} \cdot \frac{\bar{x}}{|\bar{x}|} \qquad r = |\bar{x}|$$

A unit vector in the direction of the force: $-\dfrac{\mathbf{x}}{|\mathbf{x}|}$

Inverse Square.

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$



Sound

$$k\frac{I}{r^2}$$

$$k\frac{I}{|\bar{x}|^3}$$

Notation scribbles. i,j,k-notation can help you keep track of bigger messes, organizing components into separate i,j,k-blocks, but I still think it clutters the mind.

Now maybe you see another piece of why we need to use bars over vectors in the notation. There's x-bar and then there's x.

$$\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Dude, it's the same scalar in front of each component! Pull that sucker out front!

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

$$= -\frac{mMG}{|\bar{x}|^3}\langle x, y, z\rangle$$

EXAMPLE 5 Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$q = q_1$
$Q = q_2$ } Point Charges

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

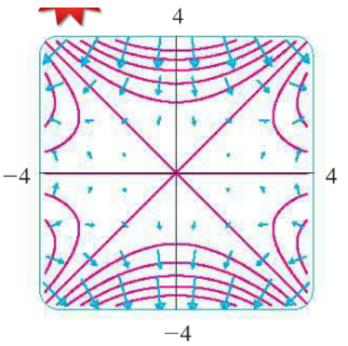
ε is in units of...

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Inverse square law in force in both gravitational and electrostatic fields. Sound also attenuates as the square of the distance from the source.

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

Gradient field overlaying contour plot.`



The surface, *f*, from Maple Session

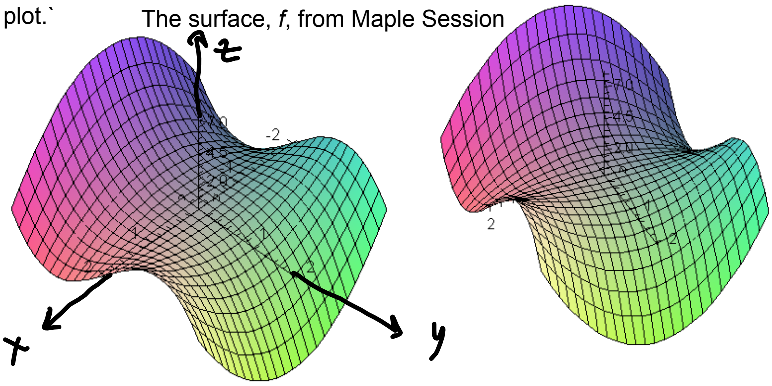


FIGURE 15

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j} = \langle f_x, f_y, f_z \rangle$$

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for **F**.

We can *build* a conservative vector field by taking the gradient of the above example...

$\bar{F}(x,y) = \langle 2xy, x^2-3y^2 \rangle$ is conservative!

Find f from $\bar{F}(x,y)$ is tougher.

$\bar{F} = \langle f_x, f_y \rangle$ if it's conservative

$f(x,y) = \int f_x\,dx = \int f_y\,dy$ is the idea.

$= \int f_x\,dx + C(y) = \int f_y\,dx + C(x)$

c(x,y), maybe   c(x,y)

" Variation of Parameters "

$\int 2xy\,dx = x^2y + C(y)$

$\int (x^2-3y^2)\,dy = x^2y - y^3 + C(x)$

or c(x,y) Memory fades, Hulk sleepy.

Now, $x^2y + C(y) = x^2y - y^3 + C(x)$

$\Rightarrow C(y) = -y^3$ & $C(x) = 0$.

$$\boxed{f(x,y) = x^2y - y^3}$$

Section 14.6 is where we learned that the gradient is always perpendicular to the contour...

The gravitational field is conservative:

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

$$= \mathbf{F}(x, y, z) = -\frac{mMG}{|\bar{x}|^3}\langle x, y, z\rangle, \text{ where } \bar{x} = \langle x, y, z\rangle$$

$$= -\frac{mMG}{|\bar{x}|^3}\,\bar{x}$$