

15.10 - Change of variable in multiple integrals.

I think we can get the idea without getting too formal about it. Just remember the tangent plane approximation to a smooth surface, and the area of a parallelogram defined by two vectors is the magnitude of their cross product.

There is no escape from u -substitution!

Here's the Chapter 4 version of Substitution:

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

$$\frac{1}{2} \int \sqrt{3x^2+2x} \cdot 2(3x+1) dx = \frac{1}{2} \int \sqrt{u} du$$

$$u = 3x^2 + 2x$$

$$du = (6x+2) dx = 2(3x+1) dx$$

$$\frac{du}{6x+2} = dx$$

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$$\int \sqrt{u} (3x+1) \frac{du}{6x+2} =$$

$$\frac{1}{2} \int \sqrt{u} du$$

In chapter 4, we were trying to see the thing on the left as something simpler, on the right, or manipulate it into something simpler, like on the right. NOW, we're going to be going from the thing on the right to the thing on the left, basically.

Substitution Rule, re-stated for Section 15.10:

By reversing the roles of x and u , we can write the Substitution Rule (4.5.5) as

$$\boxed{1} \quad \int_a^b f(x) dx = \int_c^d f(g(u))g'(u) du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$.

Another way of writing Formula 1 is as follows:

$$\boxed{2} \quad \int_a^b f(x) dx = \int_c^d f(x(u)) \frac{dx}{du} du$$

We're looking at it from the other direction, now. The motivation in Chapter 4 was to turn the thing on the right (with x 's in it) into the simpler thing on the left (with u 's instead of x 's).

Our prime example for this is polar coordinates, where we go from a function in x and y and turn it into a function of r and θ . 'ardy-'ar-D-Theta!

To convert to polar coordinates, we have the basic notion

$$x = r \cos \theta \quad y = r \sin \theta$$

and the change of variables formula (15.4.2) can be written as

$$\iint_R f(x, y) dA = \iint_S f(r \cos \theta, r \sin \theta) r dr d\theta$$

We try to generalize this idea, with polar coordinates as our touchstone.

$$T(u, v) = (x, y)$$

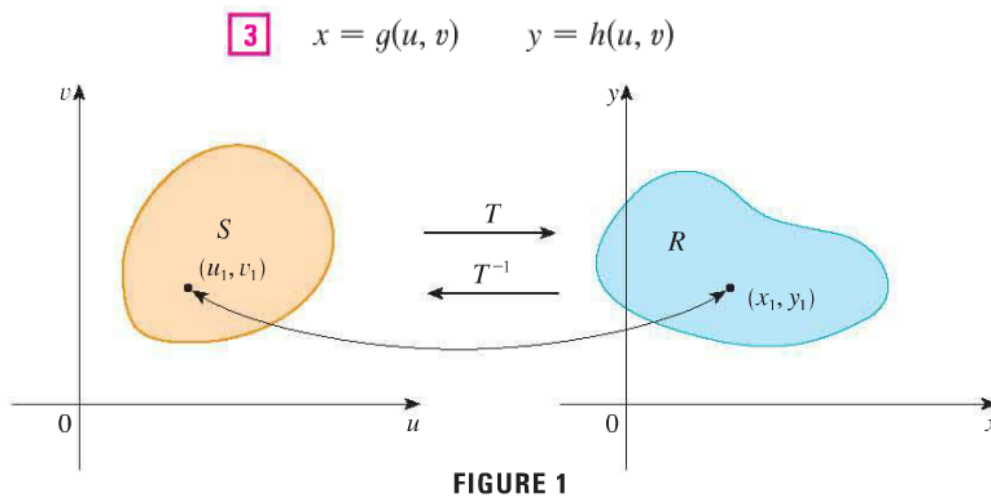
$$T(r, \theta) = (x, y)$$

$$\boxed{3} \quad x = g(u, v) \quad y = h(u, v)$$

For instance,

$$g(r, \theta) = x = r \cos \theta \quad y = r \sin \theta = h(r, \theta)$$

$$T(r, \theta) = \bar{r}(r, \theta) =$$



If T is 1-to-1, it makes sense to talk about its inverse transformation (mapping, function, ...).

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy -plane to the uv -plane and it may be possible to solve Equations 3 for u and v in terms of x and y :

$$u = G(x, y) \quad v = H(x, y)$$

EXAMPLE 1 A transformation is defined by the equations

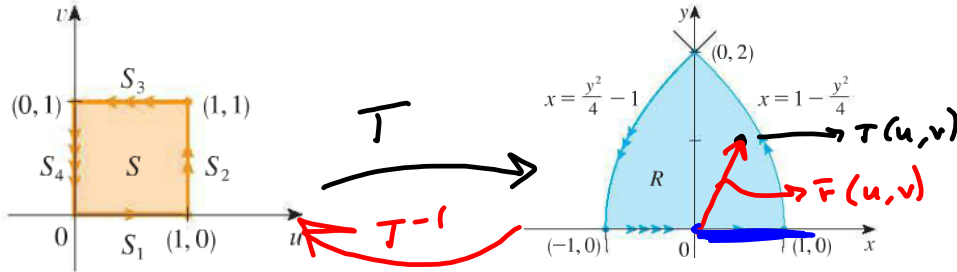
$$x = u^2 - v^2 \quad y = 2uv \quad \vec{r}(u,v) = \langle u^2 - v^2, 2uv \rangle$$

$$= T(u,v)$$

find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

Re-write in vector notation:

For practical purposes, you need to accept the statement "The boundary of the domain S is mapped to the boundary of its image, $T(S)$," and also be OK with my use of the notation $\text{Range}(T) = T(S)$.

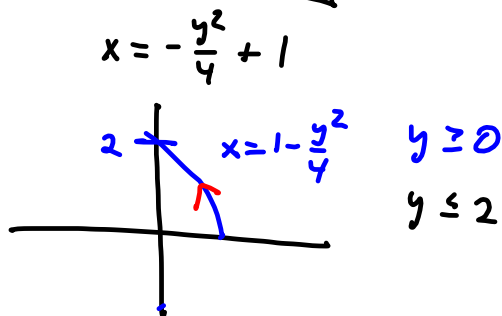
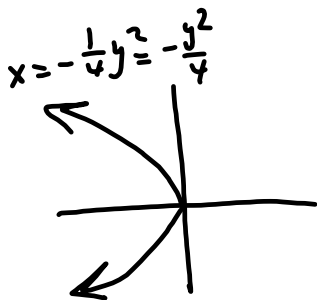
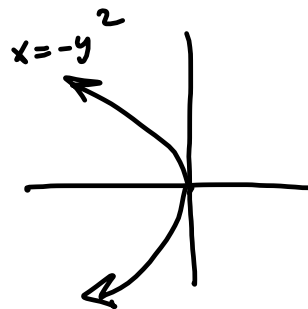
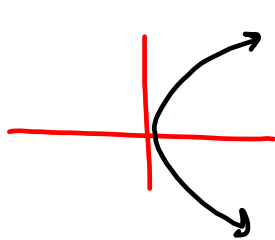


S_1
 $v=0, 0 \leq u \leq 1$
 $x = u^2 - v^2 \quad y = 2uv = 0$
 $x = u^2$
 $0 \leq x \leq 1$
 $y = 0$

In the sequel
 S_2
 $u=1, 0 \leq v \leq 1$
 $x = u^2 - v^2$
 $x = 1 - v^2$
 $y = 2uv$
 $\frac{y}{2} = v$

$0 \leq \frac{y}{2} = v \leq 1 \Rightarrow 0 \leq y \leq 2$

$x = 1 - v^2$
 $x = 1 - \left(\frac{y}{2}\right)^2$
 $x = 1 - \frac{y^2}{4}$

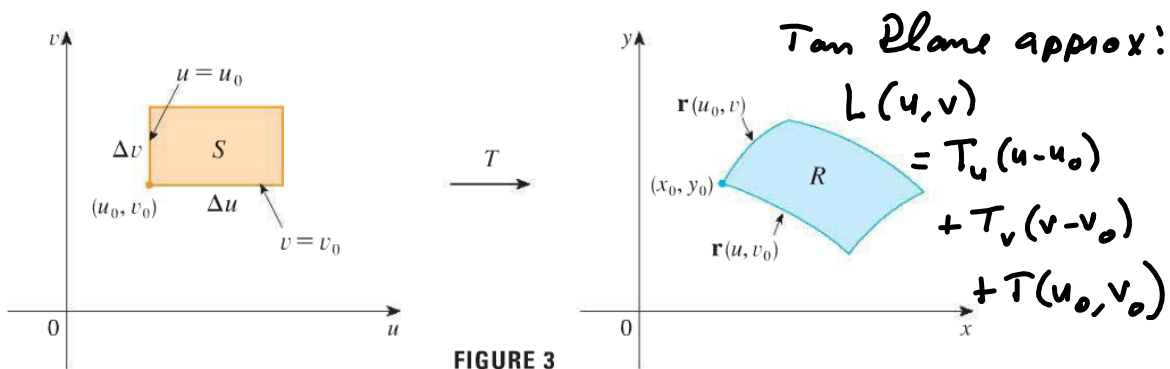


We re-write this theory in $\langle x, y \rangle$ notation, unlike the book, and try to cut to the chase.

It's really a very beautiful derivation, and it totally generalizes what we've seen specific examples of, with polar, cylindrical, and spherical coordinates. This is pretty deep stuff I still have to wrangle with.

They skirt around the fact that $T(u, v) = \mathbf{r}(u, v) = \langle x(u, v), y(u, v) \rangle$, which they also call

$\langle g(u, v), h(u, v) \rangle$, but they write them as $x(u, v) \mathbf{i}\text{-bar} + y(u, v) \mathbf{j}\text{-bar}$ or $g(u, v) \mathbf{i}\text{-bar} + h(u, v) \mathbf{j}\text{-bar}$.

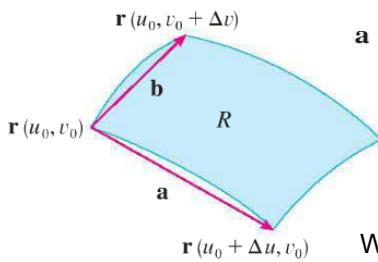


With everything in sight *smooth*, we can exploit the tangent approximation to the surface R to formulate area in the *image* space, from dA in the domain S . Very nice. We'll see that much of what we've done in this chapter is just a special case of a very general idea. May have to revisit Dr. Corliss's early take on the ar-dee-ar-dee-theta.

The trick, as always, is to come up with approximations that converge to the right thing in the limit. :o)

$$f(x, y) \approx L(x, y) = f_x(x - x_0) + f_y(y - y_0) + z_0$$

We're coming up with an expression for dA in the image.



$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \quad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

We know from example, that the image of the rectangle can be not-rectangle. But our favorite trick in calculus is how differences fade in the limit, when everything's nice and smooth.

We can approximate the image of a small rectangle by holding one, then the other variable constant, and go up one side or along another side of a nice rectangle in the uv -plane.

FIGURE 4

You see a parallelogram will be defined in the xy -plane, and it appears in the picture, to have about the same area as the image R of the rectangle S .

$$\mathbf{r}(u, v) = g(u, v) \mathbf{i} + h(u, v) \mathbf{j} = \mathbf{r}(u, v) = \langle g(u, v), h(u, v), 0 \rangle$$

$$\mathbf{r}_u = g_u(u_0, v_0) \mathbf{i} + h_u(u_0, v_0) \mathbf{j} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} = \langle g_u, h_u, 0 \rangle$$

$$\mathbf{r}_v = g_v(u_0, v_0) \mathbf{i} + h_v(u_0, v_0) \mathbf{j} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} = \langle g_v, h_v, 0 \rangle$$

Now here comes the pretty part:

Definition of the partial wrt u :

$$\mathbf{r}_u = \lim_{\Delta u \rightarrow 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u$$

This is our approximation to \mathbf{a} -bar!

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v \quad \text{Our approximation to } \mathbf{b}\text{-bar!}$$

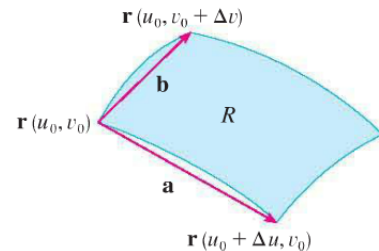


FIGURE 4

And the cross product of these two vectors will give us the area of the parallelogram living on the tangent plane!

At last! A "formula" for change of variables!

$$\boxed{6} \quad |(\Delta u \mathbf{r}_u) \times (\Delta v \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

So dA in the "image" space is given as a function of u and v , and increments in u and v directions.

To me, this equation 6 is it. Just remember that parallelogram and the cross-product of the partials times delta- u , delta- v !

The Jacobian, to me, just kinda makes it harder to remember! *sigh*

But here we go.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

I could never remember this thing, although I can remember equation 6, and remember how to do a cross product. I will absolutely hate to see somebody start with the right hand side of equation 7, even though you better have equation 7 on your cheat sheet! LOL! I'll be looking for and asking for the cross product, and expect to see it executed the shorthand way I've been showing you.

Learn it just to get points, if you have to, but it's the best way to do 'em.

$$\begin{aligned} \bar{\mathbf{r}}_u \times \bar{\mathbf{r}}_v &= \frac{\begin{vmatrix} \langle g_u, h_u, 0 \rangle \\ \langle g_v, h_v, 0 \rangle \end{vmatrix}}{\begin{vmatrix} \langle 0, 0, g_u h_v - g_v h_u \rangle \end{vmatrix}} \\ &= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \bar{\mathbf{k}} \end{aligned}$$

THIS IS OUR dA after change of variables!

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\begin{aligned} T(r, \theta) = \bar{\mathbf{r}} = \bar{\mathbf{r}}(r, \theta) &= \langle x, y \rangle \\ &= \langle r \cos \theta, r \sin \theta \rangle \end{aligned}$$

$$\bar{\mathbf{r}}_r = \langle \cos \theta, \sin \theta \rangle$$

$$\bar{\mathbf{r}}_\theta = \langle -r \sin \theta, r \cos \theta \rangle$$

$$dA = |\bar{\mathbf{r}}_r \times \bar{\mathbf{r}}_\theta| dr d\theta$$

$$\begin{aligned} \langle \cos \theta, \sin \theta, 0 \rangle \times \langle -r \sin \theta, r \cos \theta, 0 \rangle &= \langle 0, 0, r \cos^2 \theta + r \sin^2 \theta \rangle \\ &= \langle 0, 0, r \rangle \end{aligned}$$

$\bar{\mathbf{r}}_\theta d\theta \approx \text{length of } \bar{\mathbf{b}}$
 $\bar{\mathbf{r}}_r dr \approx \text{ " " } \bar{\mathbf{a}}$
in pic.

so $dA = r dr d\theta$

$0! = 1$

$$x = u^2 - v^2 \quad y = 2uv$$

$$\bar{\mathbf{r}} = \langle u^2 - v^2, 2uv \rangle$$

$$\bar{\mathbf{r}}_u = \langle 2u, 2v, 0 \rangle$$

$$\bar{\mathbf{r}}_v = \langle -2v, 2u, 0 \rangle$$

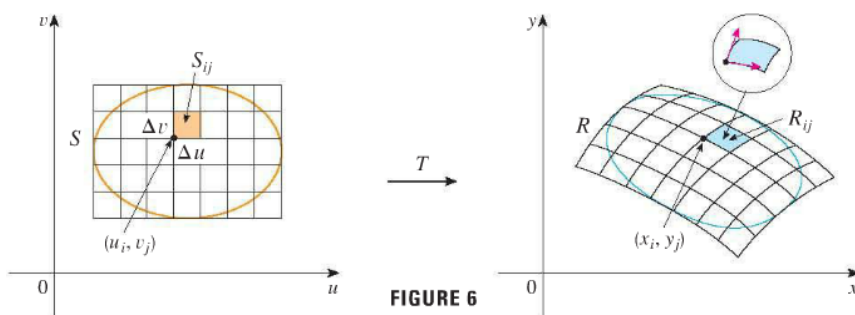
$$\langle 0, 0, 4u^2 + 4v^2 \rangle$$

$\bar{\mathbf{r}}_u \times \bar{\mathbf{r}}_v = \text{Jacobian!}$

$$\iint_S f(x, y) dx dy = \iint_R f(x(u, v), y(u, v)) (4u^2 + 4v^2) du dv$$

$$\iint_S f(x, y) dx dy = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$\boxed{8} \quad \Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$



$$\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta A \approx \sum_{i=1}^m \sum_{j=1}^n f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

In sum: **9 Change of Variables in a Double Integral** Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv -plane onto a region R in the xy -plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S . Then

$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Polar Coordinates is just a special case...

A Polar Rectangle:

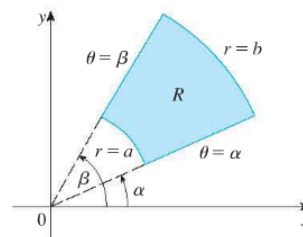
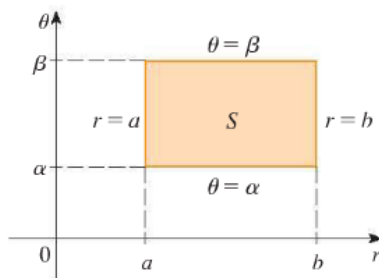


FIGURE 7
The polar coordinate transformation

$$x = g(r, \theta) = r \cos \theta \quad y = h(r, \theta) = r \sin \theta$$

It's worth seeing how the book does this. If you like my way, better, by all means. If you like the book way, better, I can probably follow what you're doing, if you observe the book formalities and don't abuse the notation.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r > 0$$

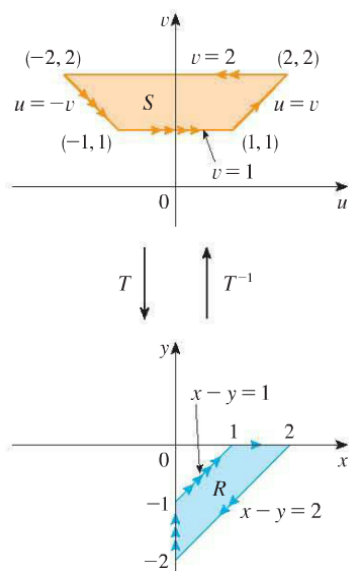
Book sorta leaves out how you get to the above determinant. With that computed, we see what we already knew:

$$\begin{aligned} \iint_R f(x, y) \, dx \, dy &= \iint_S f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta \\ &= \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \end{aligned}$$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y \, dA$, where R is the region bounded by the x -axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Not too tough, when they *hand* you the transformation. This is one we can also evaluate as the function $z = y$ over a Type II region. What's the trick in turning this into a nice square in the uv -plane?

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)} dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.



In this one, the substitution is driven by the awkwardness of the integrand.

10 $u = x + y \quad v = x - y$ This is the T -inverse

Solve the system 10 for x and y in terms of u and v :

11 $x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v)$

This is the $T(u, v) = \bar{r}(u, v)$.

To introduce the topic, we started with the region S and reverse-engineered the image $R = T(S)$. Now, we see the actual job, here is to find $T^{-1}(R) = S$.

$$\begin{array}{llll} y = 0 & x - y = 2 & x = 0 & x - y = 1 \\ u = v & v = 2 & u = -v & v = 1 \end{array}$$

$S = \{(u, v) \mid 1 \leq v \leq 2, -v \leq u \leq v\}$ Apparently type II

$$\begin{aligned} \iint_R e^{(x+y)/(x-y)} dA &= \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_1^2 \int_{-v}^v e^{u/v} \left(\frac{1}{2}\right) du dv = \frac{1}{2} \int_1^2 [ve^{u/v}]_{u=-v}^{u=v} dv \\ &= \frac{1}{2} \int_1^2 (e - e^{-1})v dv = \frac{3}{4}(e - e^{-1}) \end{aligned}$$

What the book does next is sorta unforgivable.

Recall, scalar triple product has this eerily familiar formulation as a 3x3 determinant:

$$\boxed{13} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It should come as no surprise that dV is the volume of a parallelepiped. Maybe I mis-read the text, but they never relate this to the Jacobian for Triple Integrals:

Triple Integrals. dV is sorta what you'd hope and expect, but why don't they say it?

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

$$\begin{vmatrix} g_u & h_u & k_u \\ g_v & h_v & k_v \\ g_w & h_w & k_w \end{vmatrix}$$

$$\boxed{12} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

If they wrote the TRANSPOSE of this matrix in the determinant, we could SEE that this is just extending what we did for the area of a parallelogram in the double integral to the volume of a parallelepiped in the triple integral!

Oh, and the determinant of the transpose is the same as the determinant of the original matrix.

There's nothing mysterious, here! Except, of course, how to extend this to quadruple integrals!

What's after Volume? Hypervolume?
Volume-time? (as in space-time?)

$$\boxed{13} \quad \iiint_R f(x, y, z) dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

Example 4 gives us an alternative way to view spherical coordinates, as just another special case of the general formula 13.