15.9 Triple Integrals in Spherical Coordinates

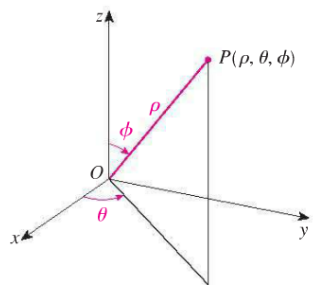


FIGURE 1
The spherical coordinates of a point

$\rho = |OP|$ is the distance from the origin to P.

ϕ is the angle between the positive z-axis and the line segment OP

$$x^2 + y^2 + z^2 = r^2$$

$$z = \pm\sqrt{r^2 - (x^2 + y^2)}$$

For situations where there's symmetry about a point, and we put that point at the origin, for convenience.

$$\begin{cases} \sqrt{r^2 - (x^2 + y^2)} \\ -\sqrt{r^2 - (x^2 + y^2)} \end{cases}$$

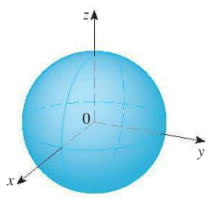


FIGURE 2 $\rho = c$, a sphere

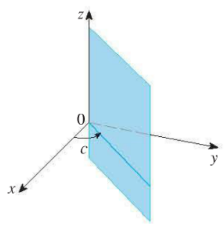
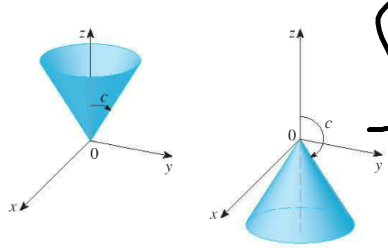


FIGURE 3 $\theta = c$, a half-plane



$0 < c < \pi/2$ $\pi/2 < c < \pi$

FIGURE 4 $\phi = c$, a half-cone

Tying spherical coords to rectangular:

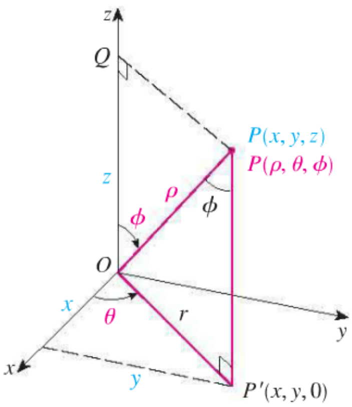


FIGURE 5

$$z = \rho\cos\phi \qquad r = \rho\sin\phi$$
$$x = r\cos\theta \qquad y = r\sin\theta,$$

This gives

$$\boxed{1} \quad x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

and, of course,

$$\boxed{2} \quad \rho^2 = x^2 + y^2 + z^2$$

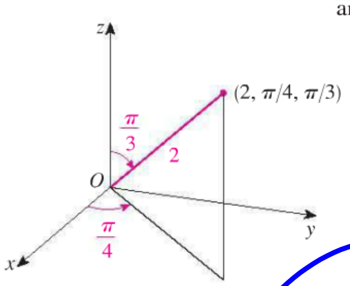
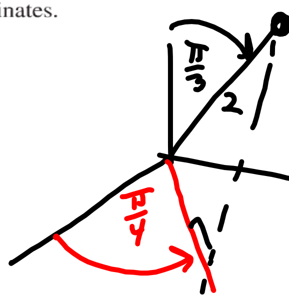


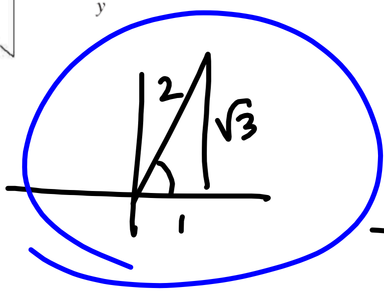
FIGURE 6

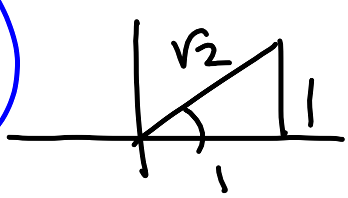
EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

(ρ, θ, ϕ)



$x = r\cos\theta$
$= \rho\sin\phi\cos\theta$
$= 2\sin\frac{\pi}{3}\cos\frac{\pi}{4}$
$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$
$= \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$





$y = r\sin\theta = \rho\sin\phi\sin\theta$
$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

$z = \rho\cos\phi$
$= 2\cos\frac{\pi}{3}$
$= 2\cdot\frac{1}{2} = 1$

$\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$

EXAMPLE 2 The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Evaluating Triple Integrals with Spherical Coordinates

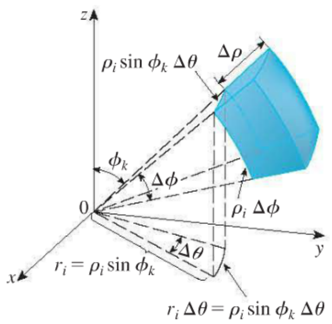


FIGURE 7

E is a chunk of a sphere...

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

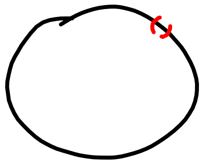
where $a \geqslant 0$ and $\beta - \alpha \leqslant 2\pi$, and $d - c \leqslant \pi$.

I think Scott will like the formulation of *dV*, with two edge lengths of our representative "rectangular box" given in terms of arc length.

$\Delta v = r\,\Delta\theta\,\rho\,\Delta\phi\,\Delta\rho$

$\Delta v = \rho^2 \sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi$

E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\,\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\,\Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$).



the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\,\Delta\phi)\underbrace{(\rho_i \sin\phi_k\,\Delta\theta)}_{\text{Maybe the hardest piece to see.}} = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$\iiint_E f(x, y, z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde\rho_i \sin\tilde\phi_k \cos\tilde\theta_j, \tilde\rho_i \sin\tilde\phi_k \sin\tilde\theta_j, \tilde\rho_i \cos\tilde\phi_k)\,\tilde\rho_i^2 \sin\tilde\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

A student prob'ly ought to write the above, once or thrice, without the asterisks and the bars. Suffice it to say those coordinates come from the *ijk* -th chunk of solid, and limit take the rest.

3 $$\iiint_E f(x, y, z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

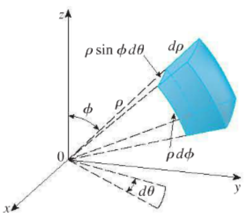
where *E* is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

So,

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$

$$\boxed{dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.}$$



Cone:
$\phi = C$
Spere
$\rho = C$

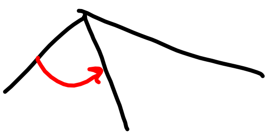
More general solid.

$$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d,\ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

The plane $x - y = 0$

$\theta = \frac{\pi}{4}$



EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

$$\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

In a tech age, the formulation and evaluation of the integral, straight-up, in rectangular coordinates is not at *all* beyond the pale. Sometimes I debate covering spherical coordinates, because a CAS has no problem chunking this ugly-looking integral:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

Type I over Type I

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int\left(\sim\right)dz\,dx\,dy$$

Type I over Type II

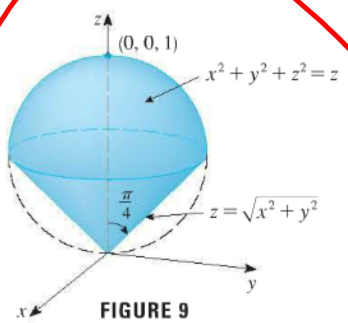


FIGURE 9

$\boxed{\text{The sphere: } x^2 + y^2 + z^2 = z.}$

$$\rho^2 = \rho\cos\phi$$
$$\rho = \cos\phi$$

The cone

$$z = \sqrt{x^2 + y^2}$$
$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,\cos^2\theta + \rho^2\sin^2\phi\,\sin^2\theta}$$
$$= \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)}$$
$$= \sqrt{\rho^2\sin^2\phi} = |\rho\sin\phi| = \rho\sin\phi \text{, b/c}$$
$$0 \le \phi \le \pi$$

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos\phi\}$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

$$x^2+y^2+z^2 = z$$
$$x^2+y^2+z^2-z+\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$
$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2$$
$$r = \tfrac{1}{2}$$
centered @ $(0,0,\frac{1}{2})$