S 15.8 #s 1, 4 - 8, 15, 18, 22, 29, 30

15.8 Triple Integrals in Cylindrical Coordinates

Recall, Polar Coordinates:

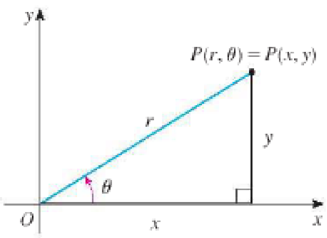


FIGURE 1

New! Cylindrical Coordinates

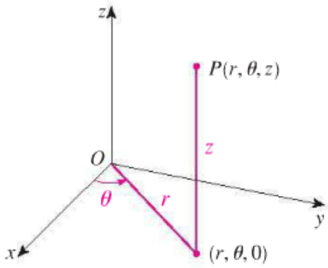


FIGURE 2
The cylindrical coordinates of a point

$$\boxed{1} \quad x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\boxed{2} \quad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

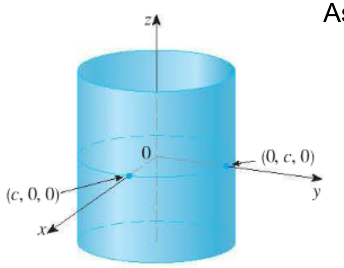


FIGURE 4
$r = c$, a cylinder

As with polar coords, representation is not unique.

$$\left(3\sqrt{2}, 7\pi/4, -7\right)$$

$$\left(3\sqrt{2}, -\pi/4, -7\right)$$

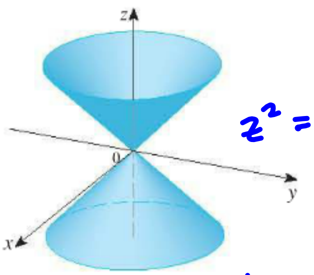
Symmetry about an axis (preferable *z* - axis) lends itself to cylindrical coordinates.

V EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

r is just the distance from a point to the *z*-axis.

$r = \sqrt{x^2 + y^2}$

$z = \sqrt{x^2 + y^2}$



$z^2 = x^2 + y^2$

$z^2 = r^2$ Yeah.

$z = \pm r$

$z = \pm\sqrt{x^2 + y^2}$

FIGURE 5
$z = r$, a cone → Meh.

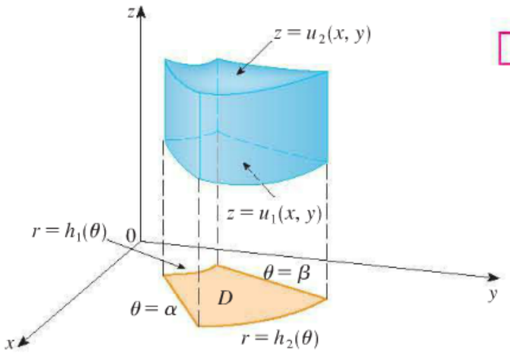
Evaluating Triple Integrals with Cylindrical Coordinates

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

Sort of a Type 1 region, then.

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

Over a domain given in polar coords.



$$\boxed{3} \quad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

See Eqn 6 in Section 15.7:

$$\boxed{6} \quad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

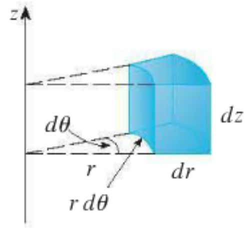


FIGURE 7
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

The dA is still $dA = r\,dr\,d\theta$, but we integrate wrt z, first, so $r\,dz\,dr\,d\theta$, not $r\,dr\,d\theta$.

Recall $r\,dr\,d\theta$?

$\frac{1}{2}r^2\theta$

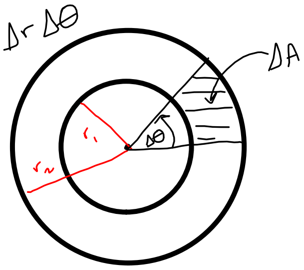
$\frac{1}{2}r^2 \Delta\theta$

$\frac{1}{2}r_2^2 \Delta\theta - \frac{1}{2}r_1^2\Delta\theta$

$= \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta$

$= \frac{1}{2}(r_2+r_1)(r_2-r_1)\Delta\theta$

$= r^* \Delta r\, \Delta\theta$



EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

lower
z-func.

$z = 1-(x^2+y^2)$
$= 1-r^2$

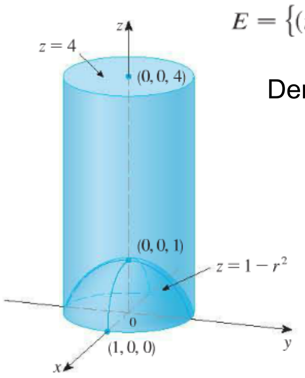


FIGURE 8

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ 1 - r^2 \leq z \leq 4\}$$

Density function proportional to distance from the z-axis.

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

density is proportional to distance from z-axis

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

Might use $\rho(x,y)$ for density func. name

$$2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

paraboloid like

$z = 1-(2x^2+y^2)$ would

not intersect cylinder

in a nice, flat circle.

EXAMPLE 4 Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

$\to g(x,y)$

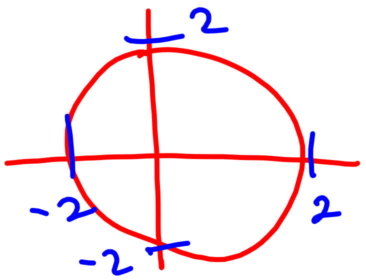
$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2 - r)\, dr$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi$$

outer 2 integrals

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} g(x,y)$$



$$= \int_0^{2\pi} \int_0^2 g(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$