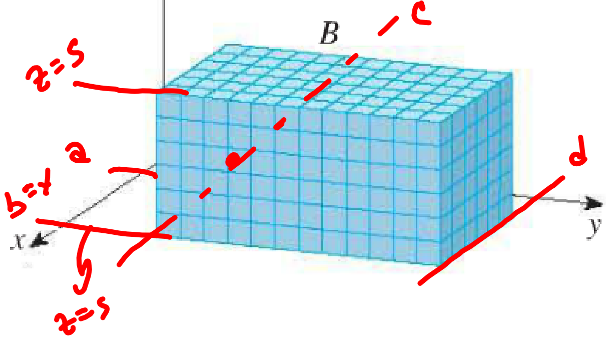
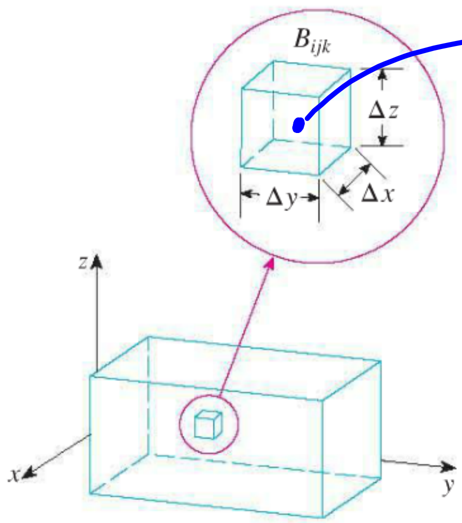
§15.7 #s 2,3,7,10,13,18,19,27,29,35,51



1 $$B = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$



$f(x_{ijk}, y_{ijk}, z_{ijk})$ is a number assigned to $(x_{ijk}, y_{ijk}, z_{ijk})$

2 $$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

4 **4 Fubini's Theorem for Triple Integrals** If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

No surprise, we evaluate over a solid by iterated integrals. Over a rectangular box, easy-peasie:

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1,\ -1 \leq y \leq 2,\ 0 \leq z \leq 3\}$$

variable
limits is goal.

Recall, Type I and Type II regions?

Adding a dimension, we now have Types 1, 2, and 3, for triple integrals, with Type 1 being the most intuitive:

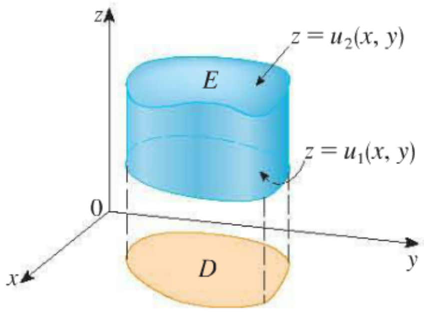


FIGURE 2
A type 1 solid region

$$\boxed{5} \quad E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

$$\boxed{6} \quad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The inside integral is a function of *x* and *y*. When we're done evaluating *it*, it's back to a double integral situation.

For this, the projection in the xy-plane decides Type I or II

The easiest variation on the easiest variation is a Type 1 solid over a Type I region in the *xy*-plane:

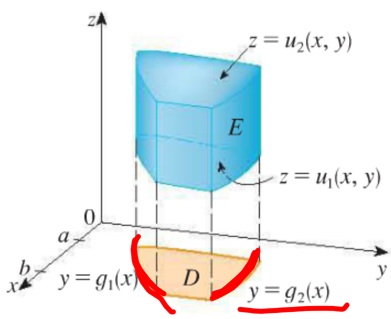
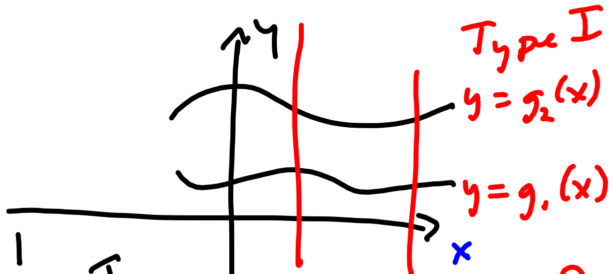


FIGURE 3
A type 1 solid region where the projection *D* is a type I plane region



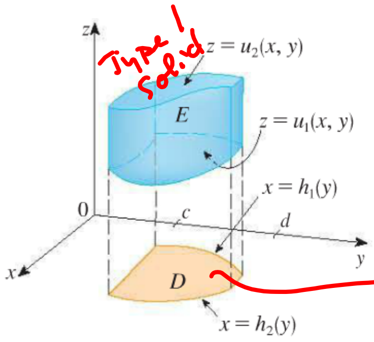
Type 1 over Type I

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \boxed{\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right)}\, dy\, dx$$

Is fun

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

$$\boxed{\boxed{7} \quad \iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$



Type II region

FIGURE 4
A type 1 solid region with a type II projection

Type 1 solid over a Type II region.

$$\boxed{8} \quad \iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

Type 1 Type II

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

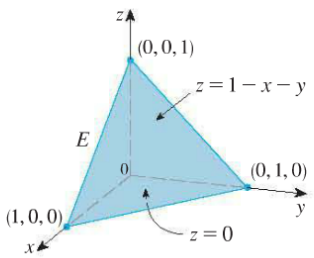


FIGURE 5

The straight-edge boundaries give us some options. This one gets formulated as Type 1 solid over a Type I projection in the xy-plane. It's not that much tougher to formulate this as a Type 1 over a Type II, since it's just as easy to solve for $x = 1 - y$ in the projection, and integrate from 0 to $1 - y$ in that middle integral, and make the outside integral from $y = 0$ to $y = 1$.

$z = 1 - x - y = f(x, y)$

$$\boxed{9} \quad E = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y\}$$

Type I

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

I think it's helpful to write out the equation 9 in your setups, and kick off the integrals with that general integral. It'll kind of be your cue on where you are, to get where you're going.

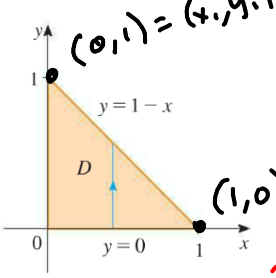


FIGURE 6

$x + y + z = 1$

In the xy-plane :

$x + y + 0 = 1$

Build it hard way

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 1}{1 - 0} = -1$

$y = m(x - x_1) + y_1$

$y = -1(x - 0) + 1$

$= -x + 1 = 1 - x$

$y = -1(x - 1) + 0$

$= -x + 1$

$= 1 - x$

$y = -x + 1$ for TI

TII : solve for x

$x = 1 - y$

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

T1 over TI

$$\int_0^1 \int_0^{1-y} \int_0^{1-x-y} z\, dz\, dx\, dy$$ T1 over TII

The next variation on the type of solid:

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

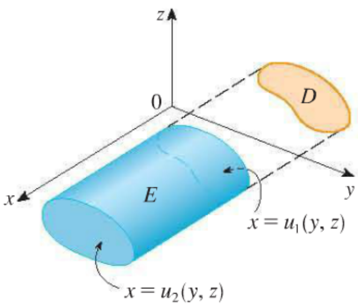


FIGURE 7
A type 2 region

$x = u_1(y, z)$ is a surface lying over yz-plane.

Integrate wrt x, 1st

$$\iint \left(\int_{\text{Back}}^{\text{front}} dx \right)$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\}$$

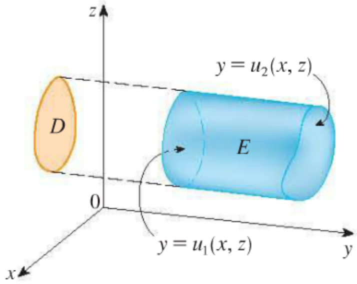


FIGURE 8
A type 3 region

$$\boxed{11} \quad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

Integrate wrt y, 1st

$$\iint \left(\int_{\text{left}}^{\text{right}} \right)$$

Warning: Video un-tested on classroom machine. *sigh*

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Debating in my mind about whether to go beyond Type 1's, since you can always re-label, to get the orientation you desire. It's what you do in the real world:

Choose the most convenient coordinate system for the application.

The author does a very nice thing, forcing a Type 1 interpretation of the solid, then recognizing it's a paraboloid on its side, treats it as a Type 3, with left function and right function over a nice disk in the *xz*-plane, and a smooth switch to polar coordinates, when he recognized the circular region, plus the nice

$$x^2 + z^2$$

in there, lookin' all like an r^2

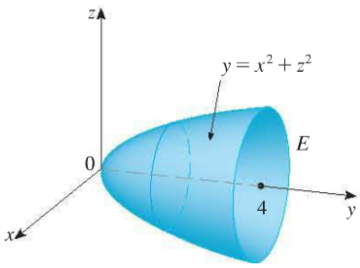


FIGURE 9
Region of integration

On a homework problem, you *really* want access to a 3-D grapher of some sort. Ideally, you can dump to a printer, and add images to homework/notes/both.

You see how it's easiest to isolate *y* so it's only natural to see this as Type 3, with a left function of *x* and *z* and a right function *y* = 4.

Again, the author force-fits it as a Type 1. That's worth seeing, and in real life, if that's the first way you see it, an ugly integral is no obstacle, with a computer's ability to crank out answers to within whatever tolerance you desire (or even symbolically precise answers).

But practicing the art form, this is Type 3 solid:

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] dA$$

$$= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

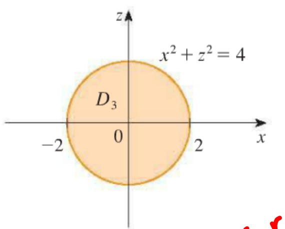


FIGURE 11
Projection onto *xz*-plane

$4 - (x^2 + z^2) = 4 - r^2$

UGH!

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx$$

$|r| = r$

$x = r\cos\theta,\ z = r\sin\theta$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, r\, dr\, d\theta = \boxed{\int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr}$$

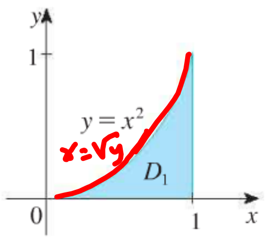
$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}$$

$f(r, \theta) = f_1(r) f_2(\theta)$

5,

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Finding what you're looking at, from the integral somebody else wrote really sends you down the rabbit-hole!

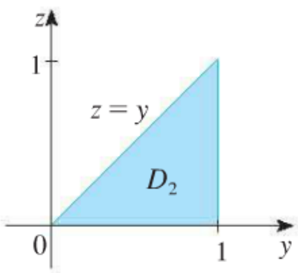


We work our way from outside-in.

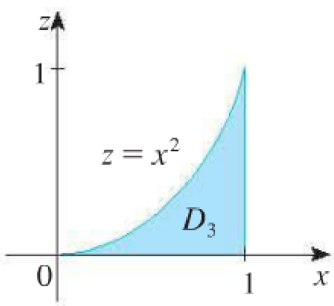
$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2, 0 \leqslant z \leqslant y\}.$$

$$D_1 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$$ ... over Type I.

$$= \left\{(x, y) \mid 0 \leqslant y \leqslant 1, \sqrt{y} \leqslant x \leqslant 1\right\}$$ ... over Type II.



$$D_2 = \{(x, y) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y\}$$



$$D_3 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant x^2\}$$

$y \leq x^2$

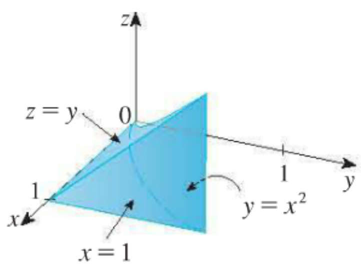


FIGURE 13
The solid E

The hard thing for me to see is the projection in the *xz*-plane, as it relates to the final 3-D picture. Hard to put myself somewhere way off to the left of this picture, looking at it from 'way down the negative *y*-axis.

To *my* eye, the "back side" of the solid, whose boundary is the cylinder

$z = x^2$, is the hardest thing to *see*. I can more easily see the cylinder

$$y = x^2$$

whose trace, you can see through the semi-transparent solid, in the *xy*-plane.

Applications

$f(x, y, z) = 1$ for all points in E.

Volume of E.

$$\boxed{12} \quad V(E) = \iiint_E dV$$

Type 1 version of volume.

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Not the textbook's usual viewing point.

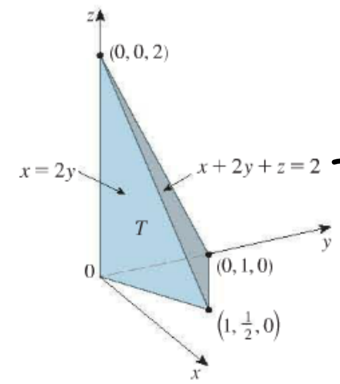


FIGURE 14

$z = 2 - x - 2y$

$$\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \int_0^{2-x-2y} dz\, dy\, dx$$

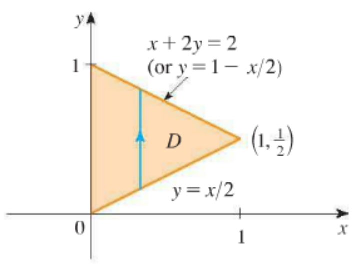


FIGURE 15

Type 1 over a Type I, makes sense.

$y = g_1(x) = \frac{1}{2}x$

$y = g_2(x) = 1 - \frac{1}{2}x$

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

I feel like this is a more satisfying way to view mass calculations. Density function, over a solid, with density per unit volume, rather than unit area, which seemed artificial.

$$\boxed{13} \quad m = \iiint_E \rho(x, y, z)\, dV$$

Moments calculated about *planes*, now.

$$\boxed{14} \quad M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

$$\boxed{15} \quad \bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

The *formulation* of these integrals matters much more than their evaluation, by hand, and it's the same art form it always was, in that regard, although you have access to way more tech than previous generations of Calc III students.

If density function is constant, the center of mass will also be the centroid (the *geometric* center of the object, basically).

Moment of inertia from
$I = mr^2$ in 1-D case

16
$$I_x = \iiint\limits_E \underbrace{(y^2 + z^2)}_{r^2} \underbrace{\rho(x, y, z)\, dV}_{m}$$

$$I_y = \iiint\limits_E (x^2 + z^2)\, \rho(x, y, z)\, dV$$

$$I_z = \iiint\limits_E (x^2 + y^2)\, \rho(x, y, z)\, dV$$

Total Charge:

As in Section 15.5, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint\limits_E \sigma(x, y, z)\, dV$$

Probability Density Function

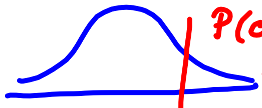
If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$P(x=1)=0$

$$P\big((X, Y, Z) \in E\big) = \iiint_E f(x, y, z)\, dV$$

$p(x,y,z)$

$E\ [a,b]\times[c,d]\times[e,f]$



$P(0 \le x \le 1)$ makes more sense

$x=1$

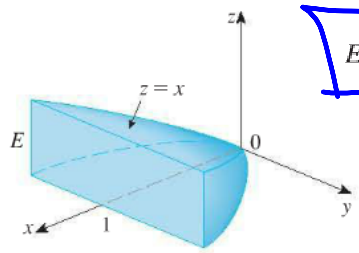
$$P(a \leq X \leq b,\ c \leq Y \leq d,\ r \leq Z \leq s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

$$f(x, y, z) \geq 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

$f(x,y,z) = cg$. Find c.

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

The work is all in the description of the region E. And the trick *there* is in seeing the paraboloid opening out of the page at you, and we're slicing it off at $z = 0$, below, and $z = x$, above.



$$\boxed{E = \{(x, y, z) \mid -1 \leq y \leq 1,\ y^2 \leq x \leq 1,\ 0 \leq z \leq x\}}$$

$$m = \iiint_E \rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\, dz\, dx\, dy$$

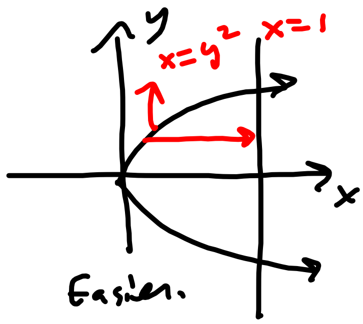
$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy$$

Once we get past writing the integral, it's of less interest. Nevertheless...

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\, dy = \rho \int_0^1 (1 - y^4)\, dy$$

$$= \rho\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

Type II



Easier.

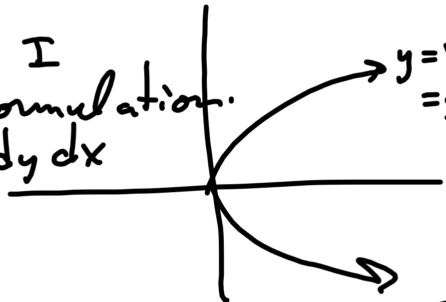
$$M_{xz} = 0$$

$$M_{yz} = \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy$$

$$M_{xy} = \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy$$

TYPE I formulation.

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int \text{stuff}\ dz\, dy\, dx$$



$dz\,dy\,dx$	1, I
$dz\,dx\,dy$	1, II
$dx\,dy\,dz$	2, I
$dx\,dz\,dy$	2, II
$dy\,dx\,dz$	3, I
$dy\,dz\,dx$	3, II

List of probs that are bothering you.