§ 15.6 #s 1, 4, 7, 10, 18, 21, 22

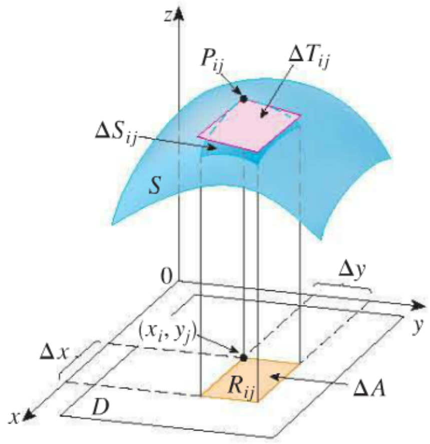


FIGURE 1

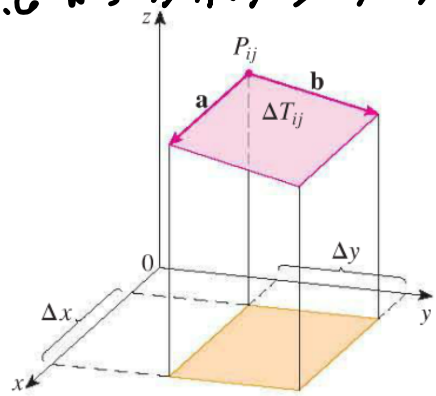


FIGURE 2

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$$

$f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of **a** and **b**

$$\mathbf{a} = \Delta x\,\mathbf{i} + f_x(x_i, y_j)\,\Delta x\,\mathbf{k} = \langle \Delta x, 0, f_x(x_i, y_i)\Delta x \rangle$$

$$\mathbf{b} = \Delta y\,\mathbf{j} + f_y(x_i, y_j)\,\Delta y\,\mathbf{k} = \langle 0, \Delta y, f_y(x_i, y_j)\Delta y \rangle$$

$$+ \quad - \quad +$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

$$\langle \Delta x, 0, f_x \Delta x \rangle$$
$$\langle 0, \Delta y, f_y \Delta y \rangle$$
$$\langle -f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y \rangle$$

$$= -f_x(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{j} + \Delta x\,\Delta y\,\mathbf{k}$$

$$|\bar{a} \times \bar{b}| = \sqrt{(f_x \Delta x \Delta y)^2 + (f_y \Delta x \Delta y)^2 + (\Delta x \Delta y)^2}$$

$$= \sqrt{(f_x^2 + f_y^2 + 1)(\Delta x \Delta y)^2}$$

$$= \sqrt{f_x^2 + f_y^2 + 1}\ \Delta x \Delta y$$

$$= \sqrt{f_x^2 + f_y^2 + 1}\ \Delta A$$

$$= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A$$

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A$$

So, put it in an integral

$\boxed{2}$ The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

$$\boxed{3} \quad A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Recall arc length, and the *ds* differential giving increment of arc length?

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now we have an incremental area.

arc length differential

$$ds = \sqrt{1 + (f')^2}\, dx$$

surface area differential

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA$$

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

$f(x,y) = x^2+y^2$

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See Figure 5.) Using Formula 3, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

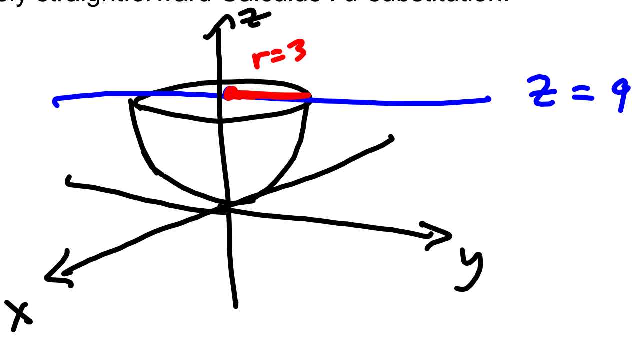
$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

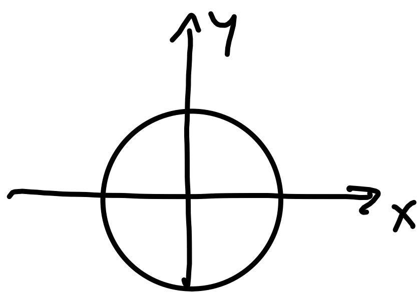
$f_x = 2x$
$f_y = 2y$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \tfrac{1}{8}\sqrt{1 + 4r^2}\, (8r)\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

As before, it's generally easier to formulate these integrals than it is to evaluate them! But this one is a relatively straightforward Calculus I *u*-substitution.





$$\int \sqrt{1+4r^2}\, r\,dr \qquad u = 4r^2+1$$

$$du = 8r\,dr$$

$$\frac{du}{8r} = dr$$

$$= \int \sqrt{u}\, \frac{du}{8r}\, r\,dr$$

$$= \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}u^{\frac{3}{2}} + C$$