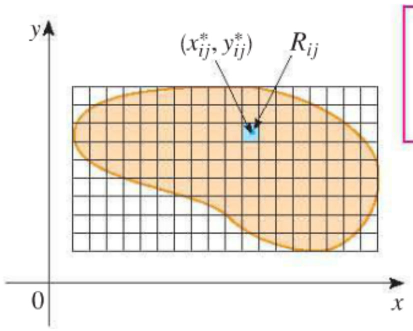
S 15.5 #s 1, 7, 16, 20, 27, 32

Mass of a lamina of uniform thickness and variable density.



$$\boxed{1} \quad m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA$$

Total charge on the surface of a plate of variable charge density.

$$\boxed{2} \quad Q = \iint_D \sigma(x, y)\, dA$$

$\sigma(x,y) = \dfrac{\text{coulombs}}{m^2}$

EXAMPLE 1 Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

SOLUTION From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}[1^2 - (1-x)^2]\, dx$$

$$= \tfrac{1}{2}\int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C.

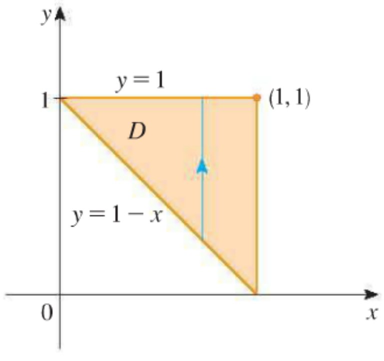


FIGURE 3

Moments and Center of Mass

Moment about the x-axis

$$\boxed{3} \quad M_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y \, \rho(x, y) \, dA$$

Moment about the y-axis

$$\boxed{4} \quad M_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x \, \rho(x, y) \, dA$$

As before, we define the center of mass $(\overline{x}, \overline{y})$ so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$.

This gives

5 The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \, \rho(x, y) \, dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \, \rho(x, y) \, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y) \, dA$$

EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$. $= z$

SOLUTION The triangle is shown in Figure 5. (Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

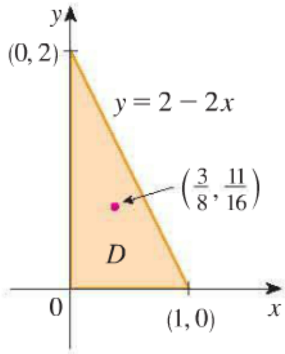
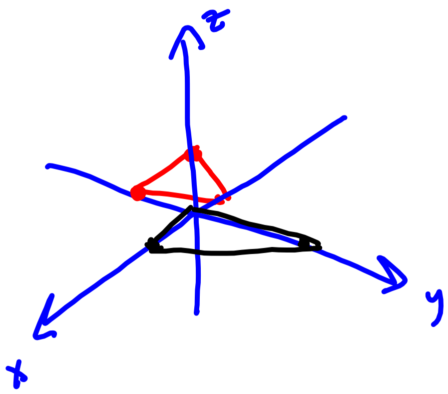


FIGURE 5



EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

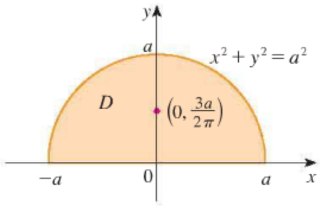


FIGURE 6

Distance is proportional to distance from the origin

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

$$\sqrt{x^2 + y^2} = r$$

$$D = \{(r,\theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq \pi\}$$

$$m = \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA$$

$$= \int_0^\pi \int_0^a (Kr)\, r\, dr\, d\theta = K \int_0^\pi d\theta \int_0^a r^2\, dr$$

$$= K\pi \frac{r^3}{3}\Big]_0^a = \frac{K\pi a^3}{3}$$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \frac{3}{K\pi a^3} \int_0^\pi \int_0^a r\sin\theta\,(Kr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3} \int_0^\pi \sin\theta\, d\theta \int_0^a r^3\, dr = \frac{3}{\pi a^3} \big[-\cos\theta\big]_0^\pi \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}$$

This gives center of mass $(0, 3a/(2\pi))$.

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments.

Moment about the x-axis

$$\boxed{6} \quad I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y^2 \rho(x, y) \, dA$$

Moment about the y-axis

$mr^2 =$ moment of inertia for a point mass.

$$\boxed{7} \quad I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x^2 \rho(x, y) \, dA$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$\boxed{8} \quad I_0 = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D (x^2 + y^2) \rho(x, y) \, dA$$

Note that $I_0 = I_x + I_y$.

Probability density functions. Meh. Main thing of interest is that you know it all adds up to 1, and the term "joint probability density function."

Book should say CONTINUOUS random variable

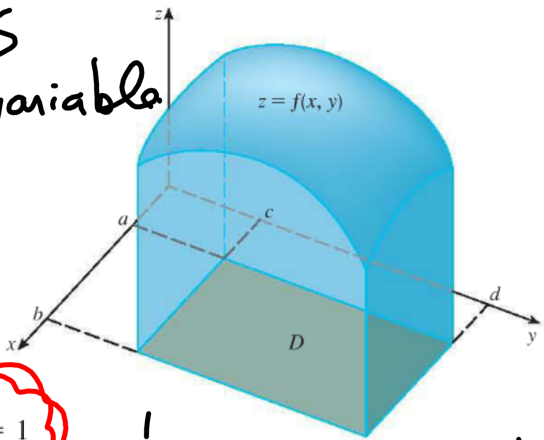
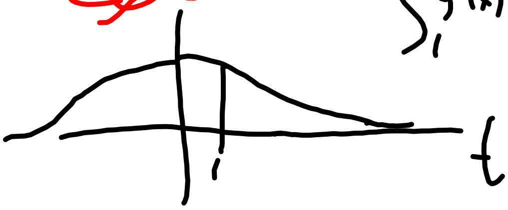


FIGURE 7

The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

$\int_1^1 f(x)\,dx = 0 = P(t) = 1$



We talk about & $P(1 \le t \le 1.5)$ is legit.

Expected Values

Recall from Section 8.5 that if X is a random variable with probability density function f, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

out come

x-mean and y-mean

probability.

11 $$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Still basically integrating a density function over a region. It's all the same thing, once you frame it, properly. It's all just moments.

Fair game?
Roll a die.

If it's a 2 or 3, I get \$20
If .. .. 1, 4, 5, I get \$30
.. .. .. 6, You get \$50

$$\left(\frac{1}{6}\right)(50) - \frac{1}{3}(20) - \frac{1}{2}(30)$$

$$\frac{50}{6} - \frac{20}{3} - 15$$

$$= \frac{50 - 40 - 90}{6} = \frac{-80}{6} = -\frac{40}{3} =$$

$$\approx -13.\overline{3}$$

Independent random variable

Notation

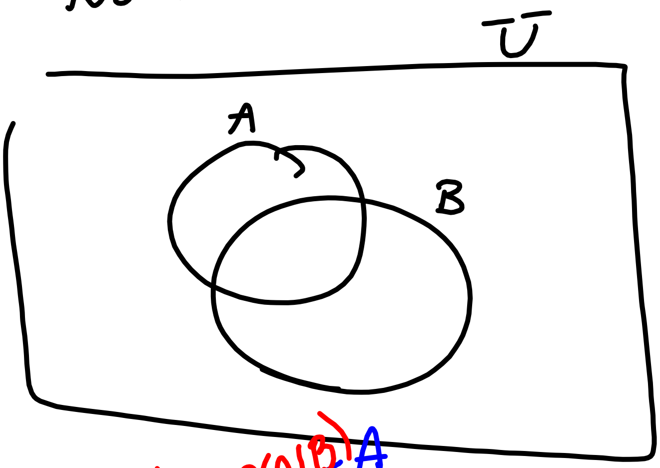
B, A = event

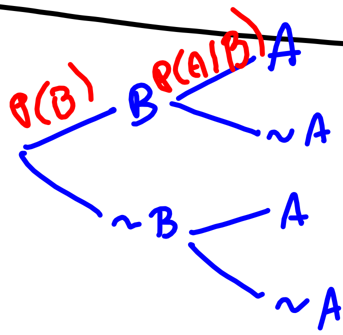
Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Independent means

$$P(A \mid B) = P(A)$$





When $P(A \mid B) = P(A)$, then

$$P(A \cap B) = P(A \mid B)\,P(B) = P(A)\,P(B)$$

See pg 1034 "Independent."