Section 15.2 - Iterated Integrals

S 15.2 #s 3k, k = 1, 2, ..., 10

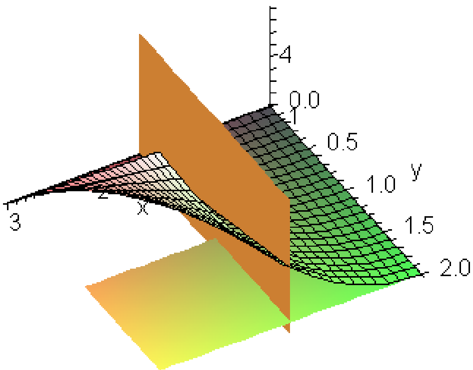
May want to tag team some of the work, if integrals get too messy.

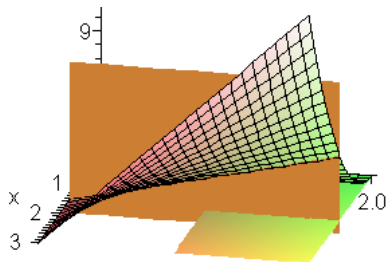
$$A(x) = \int_c^d f(x, y)\, dy$$

**EXAMPLE 1** Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Pictured is one value of $A(x)$, corresponding to $x = 1.5$. The area under the trace of the plane $x = 1.5$ in the surface. So basically, we're integrating all these area functions, for each $x$, from $x = 0$ to $x = 3$.



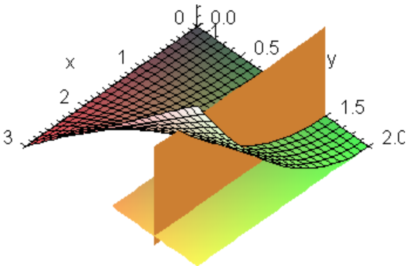

Pictures for part (b) on following page.

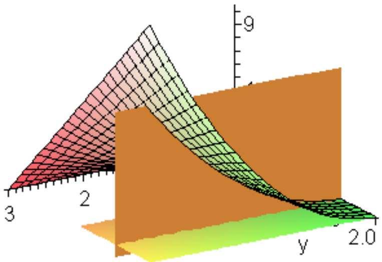
$$\int_0^3 \int_1^2 x^2 y \, dy \, dx$$

Integrating inside, 1st

$$\boxed{\int_1^2 x^2 y \, dy} = \left[ x^2 \frac{y^2}{2} \right]_{1=y}^{2=y} = \frac{1}{2}(x^2)(4-1) = \frac{3x^2}{2}$$

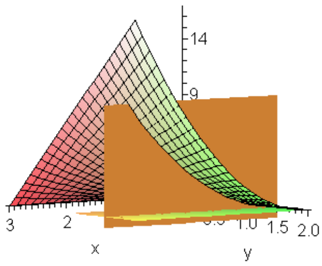
So, $\int_1^2 x^2 y \, dy$ is a function of $x$!




pictures for

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy$$




$$= \int_1^2 A(y) \, dy$$

**4 Fubini's Theorem** If $f$ is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that $f$ is bounded on $R$, $f$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

The author is pushing the envelope beyond what we need for practical purposes, but long as we're indulging ourselves, the theory of integrals goes way beyond mere finite number of discontinuities. There can even be uncountably many discontinuties, yet still have a what-you-call Legesgue-integrable function.

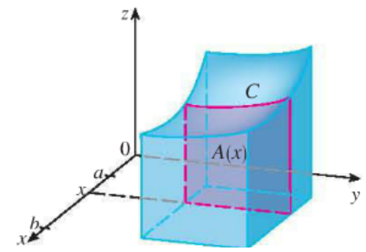
The main thing to take away, here, is that you can switch order of integration with Fubini impunity, any time you have continuity, and all bets are off when you don't have continuity (Look for denom = 0)

LOL!  Spent an hour building my own from scratch and here's this TEC link.
Click on the Earth!

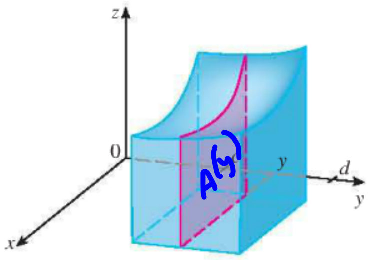
http://www.cengage.com/math/discipline_content/stewartcalc7/2008/14_cengage_tec/publish/deployments/transcendentals_7e/7e_v15_2.html

Also, the book did a way slicker job of its images.  My main advantage might be in that I can manipulate the viewing angles, and bring out the depth, if I have it "live" in my CAS front-end.

But these illustrations put mine to shame:




$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x, y)\, dy\, dx$$




$$V = \iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

A 'specially nice case.

$$f(x, y) = g(x)h(y) \quad \text{and} \quad R = [a, b] \times [c, d].$$

$$\boxed{5} \quad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

$$\int_0^3 \int_1^2 x^2 y \; dy \, dx$$

**EXAMPLE 5**

$$\iint_R \sin x \, \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy \qquad = \int_0^3 x^2 dx \int_1^2 y \, dy$$

$$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2} = 1 \cdot 1 = 1$$