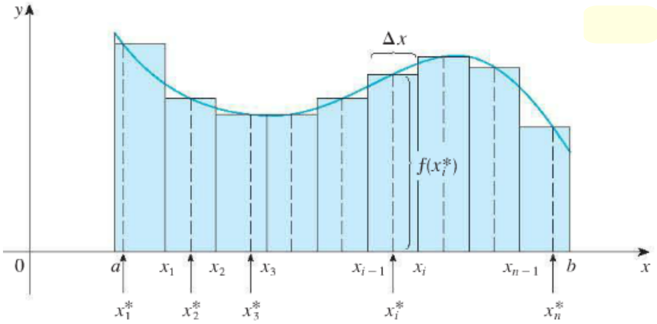
Section 15.1 Double Integrals over Rectangles

15.1 #s 17, 18. Also, pick one problem that interests you and doesn't have *too* many rectangles, unless you can find the right technology. 15.1 is about how boring a chore it is, and gee, wouldn't it be nice if we could do this more elegantly?

$$\boxed{1} \quad \sum_{i=1}^{n} f(x_i^*)\,\Delta x \xrightarrow{n \to \infty} \boxed{2} \quad \int_a^b f(x)\,dx$$

Continuity is sufficient for the limit to exist. $\Delta x = \frac{b-a}{n}$

It's not *necessary*, but how much that continuity condition can be weakened and the limit still exist is for Advanced Calculus, in particular, Measure Theory (See "Cantor Set," and proceed down the rabbit hole!)



It represents area when $f \geq 0$ on the interval in question.

Tech note: For all practical purposes, a person with the right hardware and software need never concern themselves with passing to the limit and doing calculus. Computers can whiz thru double sums (and triple sums) at the speed of light, and you can find volumes to any desired accuracy, by sampling enough data points on a surface, and adding things up.

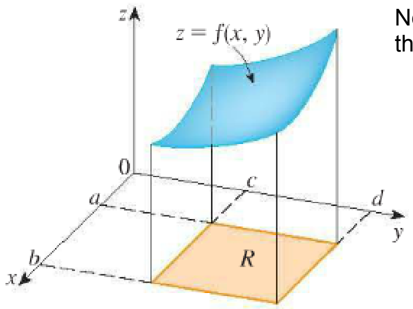


FIGURE 2

Now we bump it up a dimension and try to build the same sort of thing. Area becomes volume.

$z = f(x, y) \geq 0$

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}$$

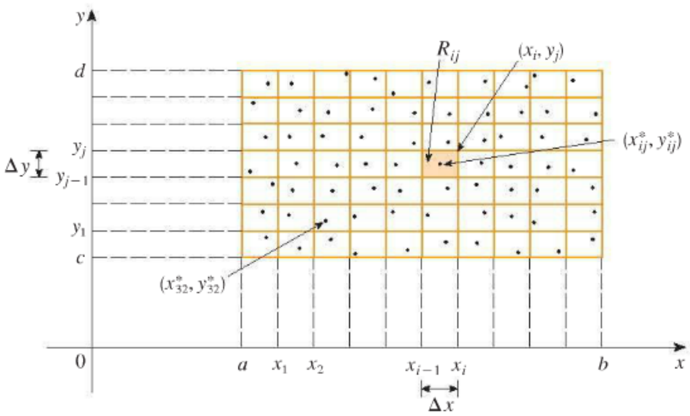
Our goal is to find the volume of S.

We divide the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$

We divide the interval $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

$\Delta A = \Delta x\, \Delta y.$ $\qquad f(x_{ij}^*, y_{ij}^*)\, \Delta A$



$$\boxed{3} \quad V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$\Delta A = \Delta y\, \Delta x = \left(\frac{d-c}{n}\right)\left(\frac{b-a}{m}\right)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\, \Delta A$$

$$= \Big(f(x_{11}, y_{11}) + f(x_{12}, y_{12}) + \cdots f(x_{1n}, y_{1n})$$
$$+ f(x_{21}, y_{21}) + f(x_{22}, y_{22}) + \cdots + f(x_{2n}, y_{2n})$$
$$+ \cdots \Big)\, \Delta A$$

$$\boxed{3} \quad V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

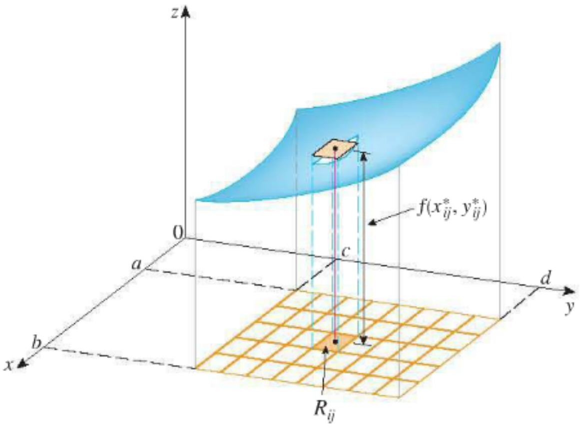


FIGURE 4

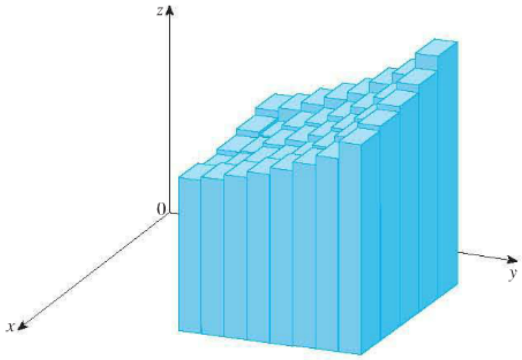


FIGURE 5

The meaning of the double limit in Equation 4 is that we can make the double sum as close as we like to the number V [for any choice of (x_{ij}^*, y_{ij}^*) in R_{ij}] by taking m and n sufficiently large.

$$\boxed{4} \quad V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$\boxed{5}$ **Definition** The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

If *f* is cnts, the limit will always exist. If *f* < 0, anywhere, it's still a legit definition of the double integral, if you're OK with "signed (negative) volume."

Formally speaking:

The precise meaning of the limit in Definition 5 is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x, y)\, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

This looks pretty Cauchy.

(Cauchy Sequences!)

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = f(c) \text{ for some } c \in (a,b).$$

MVT (for integrals) says if *f* is cnts, then you can find at least one place on the interval of integration, where *f* attains its average value. This result extends to 2-D regions in much the same way it works over intervals.

~~I'm doubting this, mostly~~ OK

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$$