14.6 Directional Derivatives, Gradient.

Click here to see an animation related to this concept.

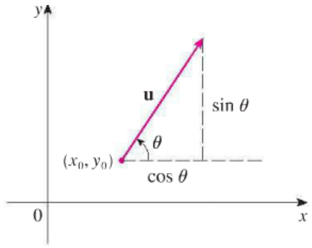
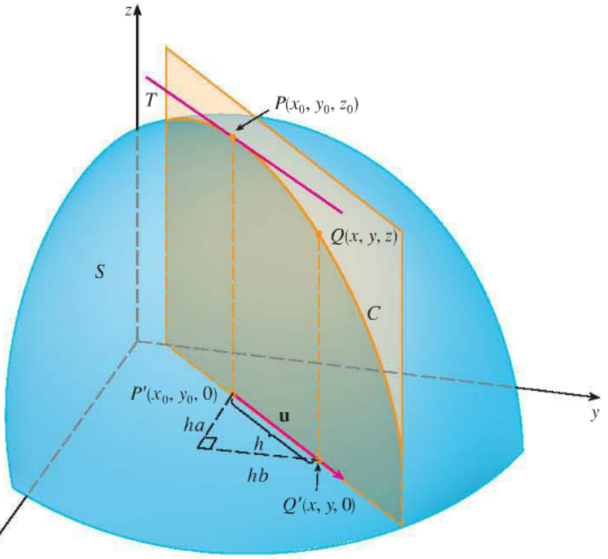


FIGURE 2
A unit vector $\mathbf{u} = \langle a, b\rangle = \langle \cos\theta, \sin\theta\rangle$



2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations [1], we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Awesome!

Make sure you convert to a *unit* vector, if you don't start with one.

The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

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$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\, a + f_y(x, y)\, b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

So $D_{\bar{u}} f(x,y) = (\nabla f) \cdot \bar{u}$

and $\nabla f(x,y) = \langle f_x, f_y \rangle$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

Don't forget that u-bar is a *unit* vector, or this is off by a factor of | u-bar |.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Just the natural extension/generalization to higher dimension.

11 $$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

Just the natural extension/generalization to higher dimension.

12 $$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\, a + f_y(x, y, z)\, b + f_z(x, y, z)\, c$$

13 $$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

14 $$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

There's **a nice visual** for this concept. TEC Visual 14.6B provides visual confirmation of Theorem 15.

Here's the basic reason *why* it's true, symbolically, although intuition is also very appealing:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

$|\bar{u}|$ gone, b/c $|\bar{u}| = 1$.

Here's a **Video to Accompany Example 6.**

EXAMPLE 6
(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

$\nabla F \cdot \bar{u} = |\nabla F||\bar{u}| \cos\theta$
is biggest when $\cos\theta = 1$!

Tangent Planes to Level Surfaces

A level surface S of a function F of 3 variables:

16 $$F(x(t), y(t), z(t)) = k$$

$P(x_0, y_0, z_0)$ be a point on S.

C be any curve that lies on the surface S and passes through the point P.

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.
$\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$.

17 $$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

18 $$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

So the gradient is perpendicular to the tangent vector at t_0 !

19 $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

is an equation of the tangent plane to the level surface!

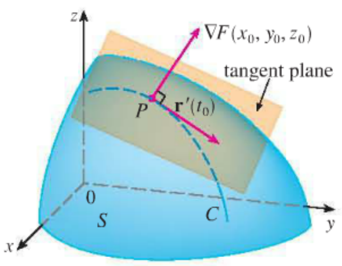

FIGURE 9