14.6 #s 1, 4, 7, 10, 13, 21, 24, 41, 44,

14.6 Directional Derivatives, Gradient.

Click here to see an animation related to this concept.

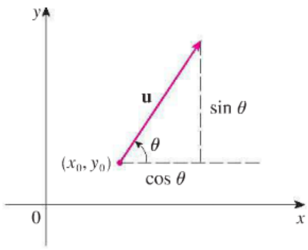
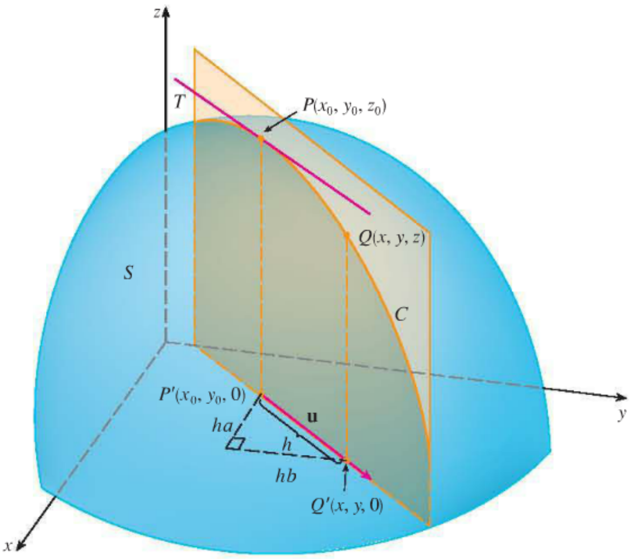



**FIGURE 2**
A unit vector $\mathbf{u} = \langle a, b\rangle = \langle \cos\theta, \sin\theta\rangle$




**2 Definition** The **directional derivative** of $f$ at $(x_0, y_0)$ in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of $f$ with respect to $x$ and $y$ are just special cases of the directional derivative.

**3 Theorem** If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Awesome!

Make sure you convert to a *unit* vector, if you don't start with one.

## The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

**7**
$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\,a + f_y(x, y)\,b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

**8 Definition** If $f$ is a function of two variables $x$ and $y$, then the **gradient** of $f$ is the vector function $\nabla f$ defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

So $D_{\bar{u}} f(x,y) = (\nabla f) \cdot \bar{u}$

and $\nabla f(x,y) = \langle f_x, f_y \rangle$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

Don't forget that u-bar is a *unit* vector, or this is off by a factor of | u-bar |.

**10 Definition** The **directional derivative** of $f$ at $(x_0, y_0, z_0)$ in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists. Just the natural extension/generalization to higher dimension.

$$\boxed{11} \quad D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

Just the natural extension/generalization to higher dimension.

**38.** The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

$V = \frac{1}{3}\pi r^2 h$

Want $\left.\frac{dV}{dt}\right|_{r=120,\ h=140}$

$\frac{dr}{dt} = 1.8 \frac{in}{s}$

$\frac{dh}{dt} = -\frac{2.5\ in}{s}$

$$\left.\frac{dV}{dt}\right|_{r=120,\ h=140} = \left[\frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}\right]_{r=120,\ h=140}$$

$$= \left[\left(\frac{2}{3}\pi r h\right)(1.8) + \left(\frac{1}{3}\pi r^2\right)(-2.5)\right]_{r=120,\ h=140}$$

$$= \frac{2}{3}\pi(120)(140)\left(\frac{9}{5}\right) - \frac{5}{2}\left(\frac{1}{3}\pi\right)(120)(120)$$

$$= \pi\left[16(1260) - 12000\right]$$

$$= \pi\left[20160 - 12000\right]$$

$$= 8160\pi$$

$1400 - 140 = 1260$

$1260 \times 16$: $7560 + 12600 = 20{,}160$