S 15.6 #s 1, 4, 7, 10, 18, 21, 22,

1–12 Find the area of the surface.

1. The part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $[0, 5] \times [1, 4]$

1. Here $z = f(x, y) = 2 + 3x + 4y$ and D is the rectangle $[0, 5] \times [1, 4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{3^2 + 4^2 + 1}\, dA = \sqrt{26} \iint_D dA$$
$$= \sqrt{26}\, A(D) = \sqrt{26}\,(5)(3) = 15\sqrt{26}$$

4. The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1)

4. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus by Formula 2,

$$A(S) = \iint_D \sqrt{1 + (3)^2 + (4y)^2}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 \sqrt{10 + 16y^2}\, \left[x \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 2y \sqrt{10 + 16y^2}\, dy = 2 \cdot \tfrac{1}{32} \cdot \tfrac{2}{3}(10 + 16y^2)^{3/2} \Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

7. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

7. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr$$

$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

10. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$

10. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x, y) \mid x^2 + y^2 \leq 3\}$ and $z = f(x, y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2 + 4 - r^2}{4 - r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

CAS **18.** Find the exact area of the surface

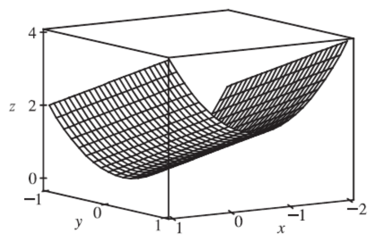
$$z = 1 + x + y + x^2 \qquad -2 \leqslant x \leqslant 1 \quad -1 \leqslant y \leqslant 1$$

18. $f(x, y) = 1 + x + y + x^2 \Rightarrow f_x = 1 + 2x,\ f_y = 1$. We use a CAS to calculate the integral

$$A(S) = \int_{-2}^{1}\int_{-1}^{1} \sqrt{f_x^2 + f_y^2 + 1}\,dy\,dx$$
$$= \int_{-2}^{1}\int_{-1}^{1} \sqrt{(1+2x)^2 + 2}\,dy\,dx = 2\int_{-2}^{1} \sqrt{4x^2 + 4x + 3}\,dx$$

and find that $A(S) = 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or

$A(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.



21. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1}\,A(D)$.

I *think* this can give you some insight on the first problem or two in this assignment.

21. Here $z = f(x, y) = ax + by + c$, $f_x(x, y) = a$, $f_y(x, y) = b$, so

$$A(S) = \iint_D \sqrt{a^2 + b^2 + 1}\,dA = \sqrt{a^2 + b^2 + 1} \iint_D dA = \sqrt{a^2 + b^2 + 1}\,A(D).$$

22. If you attempt to use Formula 2 to find the area of the top half of the sphere $x^2 + y^2 + z^2 = a^2$, you have a slight problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle $x^2 + y^2 = a^2$. However, the integral can be computed as the limit of the integral over the disk $x^2 + y^2 \leqslant t^2$ as $t \to a^-$. Use this method to show that the area of a sphere of radius a is $4\pi a^2$.

22. Let S be the upper hemisphere. Then $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, so

$$A(S) = \iint_D \sqrt{[-x(a^2 - x^2 - y^2)^{-1/2}]^2 + [-y(a^2 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$

$$= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA = \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$

$$= \lim_{t \to a^-} \int_0^{2\pi} \int_0^t \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta = 2\pi \lim_{t \to a^-} \left[-a\sqrt{a^2 - r^2}\right]_0^t = 2\pi \lim_{t \to a^-} -a\left[\sqrt{a^2 - t^2} - a\right]$$

$= 2\pi(-a)(-a) = 2\pi a^2$. Thus the surface area of the entire sphere is $4\pi a^2$.