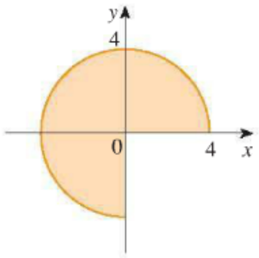
§15.4 #s 1, 4, 6, 7, 10, 13, 16, 17, 19, 22, 25, 29, 35

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

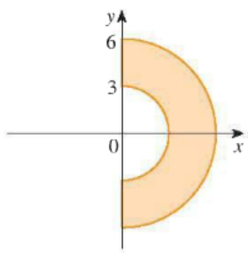
1.



1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \le r \le 4, 0 \le \theta \le \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

4.



4. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 3 \le r \le 6,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$.

Thus $\iint_R f(x, y)\, dA = \int_{-\pi/2}^{\pi/2} \int_3^6 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

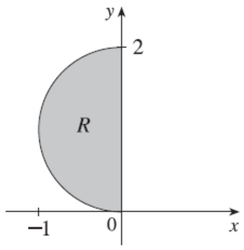
5–6 Sketch the region whose area is given by the integral and evaluate the integral.

6. $\displaystyle\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\,dr\,d\theta$

6. The integral $\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2\sin\theta,\ \pi/2 \le \theta \le \pi\}$. Since

$r = 2\sin\theta \;\Rightarrow\; r^2 = 2r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = 2y \;\Leftrightarrow$

$x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0,1)$.



$$\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$

$$= \int_{\pi/2}^{\pi} 2\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}$$

7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D x^2 y \, dA$, where D is the top half of the disk with center the origin and radius 5

7. The half disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y \, dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 (r\sin\theta)\, r \, dr \, d\theta = \left(\int_0^\pi \cos^2\theta \sin\theta \, d\theta\right)\left(\int_0^5 r^4 \, dr\right)$$

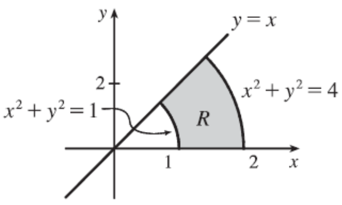
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^\pi \left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1-1)\cdot 625 = \tfrac{1250}{3}$$

10. $\iint_R \frac{y}{x^2 + y^2}\, dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

10. $$\iint_R \frac{y^2}{x^2 + y^2}\, dA = \int_0^{2\pi} \int_a^b \frac{(r \sin\theta)^2}{r^2}\, r\, dr\, d\theta = \left(\int_0^{2\pi} \sin^2\theta\, d\theta\right)\left(\int_a^b r\, dr\right)$$
$$= \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta \int_a^b r\, dr = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_a^b$$
$$= \tfrac{1}{2}\left(2\pi - 0 - 0\right)\left[\tfrac{1}{2}\left(b^2 - a^2\right)\right] = \tfrac{\pi}{2}(b^2 - a^2)$$

13. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$

13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus



$\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$.

Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

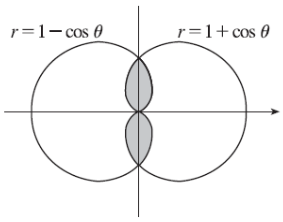
$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4} \left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64}\pi^2.$

15–18 Use a double integral to find the area of the region.

16. The region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

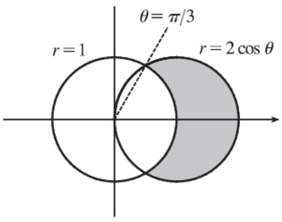
16. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1 - \cos\theta$ (see the figure). Here $D = \{(r,\theta) \mid 0 \le r \le 1 - \cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is

$$\begin{aligned}4A(D) &= 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta} d\theta \\ &= 2\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta \\ &= 2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \\ &= 2\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4}\right) = \tfrac{3\pi}{2} - 4\end{aligned}$$



17. The region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$

17. In polar coordinates the circle $(x-1)^2+y^2=1 \Leftrightarrow x^2+y^2=2x$ is $r^2=2r\cos\theta \Rightarrow r=2\cos\theta$, and the circle $x^2+y^2=1$ is $r=1$. The curves intersect in the first quadrant when $2\cos\theta=1 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\pi/3$, so the portion of the region in the first quadrant is given by $D=\{(r,\theta)\mid 1\le r\le 2\cos\theta, 0\le\theta\le\pi/2\}$. By symmetry, the total area is twice the area of D:



$$2A(D)=2\iint_D dA=2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta=2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta}d\theta$$
$$=\int_0^{\pi/3}\left(4\cos^2\theta-1\right)d\theta=\int_0^{\pi/3}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta)-1\right]d\theta$$
$$=\int_0^{\pi/3}(1+2\cos 2\theta)\,d\theta=\left[\theta+\sin 2\theta\right]_0^{\pi/3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

19–27 Use polar coordinates to find the volume of the given solid.

19. Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leqslant 4$

19. $V = \iint_{x^2+y^2 \le 4} \sqrt{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2\, dr = \left[\,\theta\,\right]_0^{2\pi} \left[\frac{1}{3} r^3\right]_0^2 = 2\pi\left(\frac{8}{3}\right) = \frac{16}{3}\pi$

22. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

22. The sphere $x^2 + y^2 + z^2 = 16$ intersects the xy-plane in the circle $x^2 + y^2 = 16$, so

$$V = 2 \iint_{4 \le x^2+y^2 \le 16} \sqrt{16 - x^2 - y^2}\, dA \quad \text{[by symmetry]} \quad = 2 \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\; r\, dr\, d\theta = 2 \int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} dr$$

$$= 2\big[\,\theta\,\big]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0 - 12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

25. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

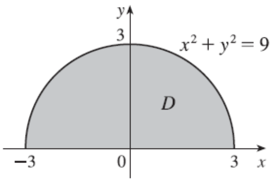
25. The cone $z = \sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$ or $x^2+y^2=\frac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r\, dr\, d\theta$$

29–32 Evaluate the iterated integral by converting to polar coordinates.

29. $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(x^2 + y^2)\, dy\, dx$

29.



$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)dy\,dx = \int_{0}^{\pi} \int_{0}^{3} \sin\left(r^2\right) r\,dr\,d\theta$$
$$= \int_0^\pi d\theta \int_0^3 r \sin\left(r^2\right) dr = [\theta]_0^\pi \left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3$$
$$= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1 - \cos 9)$$

35. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

35. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x, y)$ to be the depth of the water at (x, y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x, y) \mid x^2 + y^2 \leq 400\}$ and below the graph of $f(x, y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0, -20) = 2$ to $f(0, 20) = 7$. The trace in the yz-plane is a line segment from $(0, -20, 2)$ to $(0, 20, 7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \quad \Rightarrow \quad z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x, y)$ is independent of x, $f(x, y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x, y)\, dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r, \theta) \mid 0 \leq r \leq 20, 0 \leq \theta \leq 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$

$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$